Geometry introduction

Geometry Goals

- Understand how results (theorems) are built from assumed properties (axioms) in mathematics.
- Be able to prove theorems in Euclidean geometry
- Understand the foundational theorems in Euclidean geometry
- Understand some of the results of making different assumptions about basic properties in a geometric system (in particular in the historically significant cases of spherical and hyperbolic geometry).

Why these goals?
Answer 1: Educational Theory
Dina and Pierre Van Hiele

Lv 0: Visualization ("it looks like a square")
Lv 1: Analysis ("the square has 4 sides and 4 right angles")
Lv 2: Abstraction ("squares have 4 right angles, so any square is also a rectangle")
Lv 3: Deduction (can construct simple proofs)
Lv 4: Rigor (can construct some more complex proofs, and understands geometry as an axiomatic system, and compares axiomatic systems)

Why these goals?
Answer 2: This is what mathematics "is"

For a mathematician, the most important part of mathematics is deduction: proving that new result must be true if we accept certain basic assumptions.

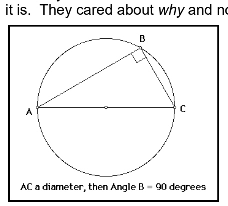
The current form of how math is built was laid out systematically in the late 1800s, when mathematicians carefully explored:

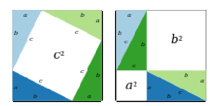
- Set theory--what we can (and can't) deduce just from sets
- The real number system (ties into infinite sets)
- Limits (how can we use real numbers to define limits)
- Geometry (what are *all* the assumptions we need to get Euclidean geometry)

Why these goals?
Answer 3: History

Geometry made mathematics what it is today!

~600 BC Thales of Miletus and other Greeks of his time studied math and one of the things they did (the most influential thing) was they studied/discussed/valued knowing *why* math is the way it is. They cared about *why* and not just *what*.





Pythagoras studied with Thales and founded a school that studied mathematics this way (as well as other things)

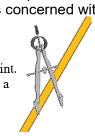
Why these goals?
Answer 3: History

Geometry made mathematics what it is today!

~300 BC Euclid wrote the *Elements* which organized the proofs in geometry in a linear (and axiomatic) way, where proofs were built on previously proven results. (We don't know how much of the results and proofs were original to Euclid, but he was the master of the organization).

The Elements was the basic text for geometry until about 1900.

~300 BC Most of Euclid's geometry axioms (postulates) looked very different from what axioms look like today, but they were important for his audience because they defined that his system was concerned with circle and straight-edge constructions



Postulates
<u>Postulate 1</u>. **To draw a straight line** from any point to any point.
<u>Postulate 2</u>. **To produce a finite straight line** continuously in a straight line.
<u>Postulate 3</u>. **To describe a circle** with any center and radius.
<u>Postulate 4</u>. That all right angles equal one another.

<u>Postulate 5</u>. That, **if** a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, **the two straight lines**, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Geometry introduction

Why these goals?
Answer 3: History

Mathematicians at first didn't like the complicated 5th axiom because it didn't look simple enough. They tried to prove the 5th axiom result using just the first 4 axioms, and ended up discovering a lot of different conditions that are equivalent to the 5th axiom: you can change the 5th axiom, but you can't get rid of it:

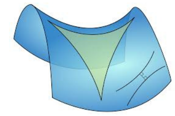
~ 2nd c AD Ptolemy (also Playfair. ~1800) *There is at most one line that can be drawn parallel to another given one through an external point.*
~ 5th c AD Proclus *Points on parallel lines stay a bounded distance apart.*
~ 9th c AD al-Gauhary *From any point in an angle it is possible to draw a line that intersects both sides of the angle*
1663 AD John Wallis *It is possible to enlarge a triangle without distorting the angles.*

Why these goals?
Answer 3: History

~ 1830 Nikolai Lobachevsky and Janos Bolyai (and Gauss) independently proved that if you change the 5th postulate, you get a consistent geometric system that's different from Euclidean geometry (hyperbolic geometry)

Euclidean geometry: *There is* ***at most one*** *line that can be drawn parallel to another given one through an external point.*

Hyperbolic geometry: *There is* ***more than one*** *line that can be drawn parallel to another given one through an external point.*



Why these goals?
Answer 3: History

Geometry made mathematics what it is today!

All this work and thinking about the 5th postulate made mathematicians better at identifying subtle assumptions and led them to conclude that axioms should look more like the 5th postulate than the first 4, and we probably need more of them

Postulates
Postulate 1. **To draw a straight line** from any point to any point.
Postulate 2. **To produce a finite straight line** continuously in a straight line.
Postulate 3. **To describe a circle** with any center and radius.
Postulate 4. That all right angles equal one another.

Postulate 5. That, **if** a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, **[then]** the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Why these goals?
Answer 3: History

Geometry made mathematics what it is today!

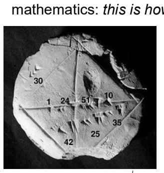
~19th c AD Mathematicians made axiomatic systems for lots of areas of mathematics. They were trying to make sure that mathematical reasoning didn't leave any gaps that could lead to errors and inconsistencies. A lot of math results were reproven in a more careful axiomatic system to make sure everything was complete. Important math foundations topics:

- logic
- set theory
- geometry
- abstract algebra
- analysis (calculus)

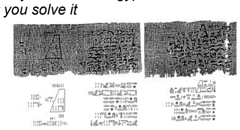
Learning from our "mistakes" in geometry* transformed mathematics.

*There are others, but none so widely debated and at such great length as the parallel postulate question.

The way we teach and learn math in earlier grades is sometimes most similar to how the Babylonians and Egyptians recorded their mathematics: *this is how you solve it*

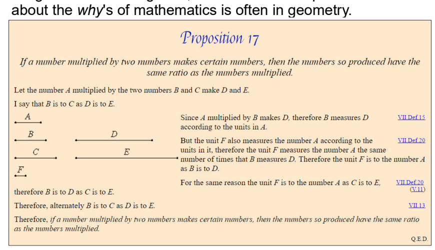


Approximation of √2
~1600 BC?
1.41421296...



How to find the volume of a truncated pyramid
~1800 BC

The Greeks, by contrast, wanted to know and show *why* things worked. In part because they didn't have a robust enough number system, they relied on geometry to show why things worked, even things about arithmetic.
Geometric pictures are good for conveying information about how things work and fit together, so one of the first places we learn about the *why*'s of mathematics is often in geometry.



Proposition 17

If a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.

Let the number *A* multiplied by the two numbers *B* and *C* make *D* and *E*.

I say that *B* is to *C* as *D* is to *E*.

Since *A* multiplied by *B* makes *D*, therefore *B* measures *D* according to the units in *A*. VII.Def.15

But the unit *F* also measures the number *A* according to the units in it, therefore the unit *F* measures the number *A* the same number of times that *B* measures *D*. Therefore the unit *F* is to the number *A* as *B* is to *D*. VII.Def.20

For the same reason the unit *F* is to the number *A* as *C* is to *E*, therefore *B* is to *D* as *C* is to *E*. VII.Def.20 (V.11)

Therefore, alternately *B* is to *C* as *D* is to *E*. VII.13

Therefore, *if a number multiplied by two numbers makes certain numbers, then the numbers so produced have the same ratio as the numbers multiplied.*

Q.E.D.

Geometry introduction

In this class we're going to work on:

- Geometry, and why it works
- What axioms are
- How changing an axiom (especially the parallel postulate) affects everything else
- How to prove results (theorems) from assumptions (axioms)
- How to read a geometry statement (e.g. a postulate or theorem from Euclid's axioms)
- How to write a statement that means what you intend for it to mean
- What the foundational theorems are and why they are true

- **How to read a geometry statement (e.g. a postulate or theorem from Euclid's axioms)**

Postulates

Postulate 1. **To draw a straight line** from any point to any point.

Postulate 2. **To produce a finite straight line** continuously in a straight line.

Postulate 3. **To describe a circle** with any center and radius.

- **How to read a geometry statement (e.g. a postulate or theorem from Euclid's axioms)**

Propositions

Proposition 1.
To construct an equilateral triangle on a given finite straight line.

Proposition 2.
To place a straight line equal to a given straight line with one end at a given point.

Proposition 3.
To cut off from the greater of two given unequal straight lines a straight line equal to the less.

- **How to read a geometry statement (e.g. a postulate or theorem from Euclid's axioms)**

Propositions

Proposition 9.
To bisect a given rectilinear angle.

Proposition 10.
To bisect a given finite straight line.

Proposition 11.
To draw a straight line at right angles to a given straight line from a given point on it.

Proposition 12.
To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

- **How to read a geometry statement (e.g. a postulate or theorem from Euclid's axioms)**

Propositions

Proposition 22.
To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition 23.
To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.