Geometry Background Knowledge.

In Chapter 1, we will be using some Analytic Geometry to show some properties. Analytic Geometry is the system of describing points, lines, planes, circles, etc. using coordinates and equations. Both standard and parametric equations are allowed tools for analytic geometry.

Some prior knowledge that you should look up if you do not remember it from your precalculus, calculus and linear algebra classes:

- **Formula for the slope** of a line in the plane
- Finding an equation of a line using the **point-slope formula** for a non-vertical line
- Finding an **equation of a vertical line**.
- Finding an equation for a **parallel line**
- Finding an equation for a **perpendicular line**
- Alternate: finding an equation of a line using two points in a vector representation. (**formula for vector equation for a line)**
- Finding an **equation of a plane** using three non-collinear points in a **vector representation**.
- Converting an **equation of a plane** from vector representation to **standard form:** $ax+by+cz=d$ (where at least one of the constants *a, b, c* is non-zero)
- Finding an **equation of a circle using a center and a point** on the circle
- Finding the **reflection of a point across the x-axis** in the plane.

Affine Geometry:
There exists a set of objects called Points
There exists another set of objects called Lines. Each Line is a set that includes some Points.
If there is a Point and a Line then either the Point is in/on the Line or the Point is not in/on the Line.

1. For every two distinct Points there is exactly one Line that contains both of them
2. **Any time there is a Line, and a Point that is not in/on that Line, then there is another Line that contains the Point, and does not intersect with the first Line.**

Projective Geometry:
There exists a set of objects called Points
There exists another set of objects called Lines. Each Line is a set that includes some Points.
If there is a Point and a Line then either the Point is in/on the Line or the Point is not in/on the Line.

1. For every two distinct Points there is exactly one Line that contains both of them
2. **Any time there are two distinct Lines, then there is exactly one Point that is in both Lines**.

Homework problems:

1. Show that in the x-y coordinate plane, if you have any two distinct points (s,t) and (u,v), then you can write an equation of a line that includes both of the points. Do this using two cases:

  a. Case 1: $s=u$

  b. Case 2: $s \neq u$

1c. What does 1a and 1b prove? (Through each pair of points there exists exactly one line? At least one line? At most one line?

2. Given a point (a,b) that does not lie on the line through (s,t) and (u,v), show how to get the equation of a line that includes (a,b), and does not intersect the line through (s,t) and (u,v). Do this using the cases:

  a. Case 1: $s=u$ (use the equation from 1a)

  b. Case 2: $s \neq u$ (use the equation from 1b)

3. Problems 1 and 2 provide evidence that the x-y coordinate plane is a(n) _____________ geometry.

4. Given that (a,b,c) is a point on the unit sphere with center at the origin, tell the coordinate for its antipode, and show that its antipode point lies on the line through the origin: $(x,y,z)=t(a,b,c)$

5. Given that (a,b,c) and (d,e,f) are points in space where (d,e,f) does not lie on the line $(x,y,z)=t(a,b,c)$, write the equation of the plane that includes $(0,0,0)$, (a,b,c) and (d,e,f)

  a. in vector form

  b. in standard form.

6. Note that in standard form, a plane through the origin is of the form $ax+by+cz=0$. Show that if (x,y,z) (a point on the unit sphere) lies on a plane that includes $(0,0,0)$, then the antipodal point to (x,y,z) lies on the same plane.

Definition: A great circle is the intersection of a sphere with a plane that passes through the center of the sphere.

7. Explain how the previous problems show that: if (a,b,c) and (d,e,f) are points in the unit sphere, where (d,e,f) does not lie on the line $(x,y,z)=t(a,b,c)$, then the points (a,b,c), (d,e,f) and their antipodal points all lie on the same great circle.

Definition: The system where Points (or P-points) are defined as pairs of antipodal points on the unit sphere, and Lines (or P-Lines) are defined as great circles on the unit sphere is called the Projective Plane.

8. Which of these did we show are true in problems 4-7: Through every pair of P-points, there exists a unique P-line? At least one P-line? At most one P-line?

9. Match each fact about 3D Analytic Geometry with the fact it would prove about the Projective Plane:

a. Through any three non-collinear points, there exists a unique line	I. Latitudes are not P-lines
b. Every pair of distinct planes through the origin intersect in a line through the origin.	II. Through any pair of distinct P-points there is a uniqe P-line
c. The plane $z=.5$ does not pass through the origin.	III. Every pair of distinct P-lines intersects in a P-point.

10. Problems 4-9 provide evidence that the Projective Plane is a(n) _____________ geometry.

11. Given points (a,b) and (c,d) where $b>0$, $d>0$ and $a \neq c$, describe how to find the correct x-coordinate for a circle whose center lies on the x-axis and that includes both points (a,b) and (c,d)

12. Given points (a,b) and (c,d) where $b>0$, $d>0$ and $a=c$, tell what the equation of a vertical line that includes both points would be.

Definition: The Upper Half Plane model for hyperbolic geometry consists of Points in the upper half of the x-y plane, eg. (a,b) where $b>0$, which I will write U-points, and Lines which include: vertical lines (just the part in the half-plane) and half-circles where the center of the circle lies on the x-axis; I will write U-lines as a way of including both the vertical lines and semicircles.

13. Which of these did we show are true in problems 11-12: Through every pair of U-points, there exists a unique U-line? At least one U-line? At most one U-line?

14. a. If you have a pair of points (a,b) and (c,d) where $b>0$, $d>0$ and $a \neq c$, is it possible to find a vertical line between them?
b. If you have a pair of points (a,b) and (c,d) where $b>0$, $d>0$ and $a=c$, is it possible for them to both be on a circle where the center of the circle lies on the x-axis?

15. What do the answers to 14 say about U-lines? Through every pair of U-points, there exists a unique U-line? At least one U-line? At most one U-line?

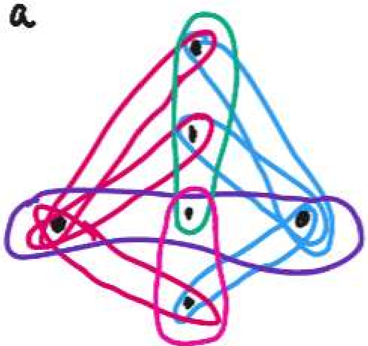
16. If you have a vertical line in the plane: $x=h$ and a point (a,b) where $b>0$ that does not lie on that vertical line, how could you find a U-line that would go through (a,b) that doesn't intersect $x=h$?

17. If you have a circle with center on the x-axis: $(x-h)^2+y^2=r^2$ and you have a point (a,b) where $b>0$ that does not lie on that circle, how could you find a U-line that would go through (a,b) that doesn't intersect $(x-h)^2+y^2=r^2$?

18. Problems 11-17 provide evidence that the Upper-Half-Hyperbolic Plane is a(n) _____________ geometry.

19. More practice questions: At least one of these examples is an affine geometry, at least one is a projective geometry, and at least one does not satisfy the unique-lines axiom. Explain what category each belongs in and why

a



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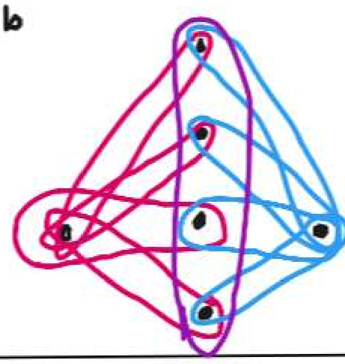

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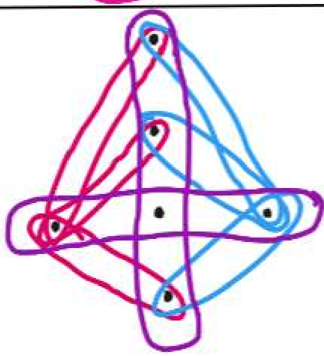

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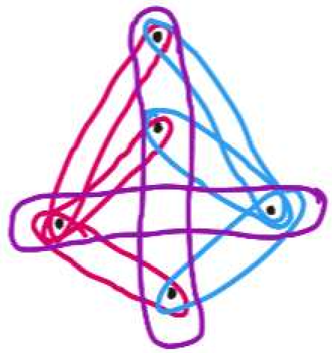

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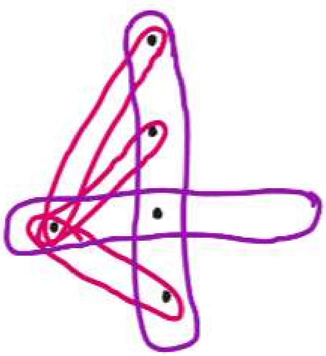

f. Points A, B, C, D, E, F, G
Lines: ABC, CDE, AEF,
AGD, CGF, BEG
BDF