Theorem 20: Given segments $\overline{AB}$ and $\overline{CD}$ with equal lengths: $\underline{AB} = \underline{CD}$, there exists an isometry $f : E^2 \to E^2$ such that $f(A) = C$ and $f(B) = D$, and hence $\overline{AB} \cong \overline{CD}$.

Proof:

Given $\overline{AB}$ and $\overline{CD}$ such that $\underline{AB} = \underline{CD}$

By Axiom 3, there is an isometry f such that

- $f(A) = C$
- $f(B) \in \overrightarrow{CD}$

Because f is an isometry, we know $\underline{AB} = \underline{f(A)f(B)}$ and so $\underline{AB} = \underline{Cf(B)}$

By substitution (into $\underline{AB} = \underline{CD}$) we get $\underline{CD} = \underline{Cf(B)}$

By definition $\overrightarrow{CD} = \{X \mid \underline{CX} + \underline{XD} = \underline{CD} \text{ or } \underline{CD} + \underline{DX} = \underline{CX}\}$

Because $f(B) \in \overrightarrow{CD}$, we know $\underline{C\,f(B)} + \underline{f(B)D} = \underline{CD}$ or $\underline{CD} + \underline{Df(B)} = \underline{C\,f(B)}$

> This is the question/clarification. If we are going to write $\overrightarrow{CD} =$ then we technically need a variable (like X) in the definition.
>
> It's also OK to compress these two lines into one line and say:
>
> Because $f(B) \in \overrightarrow{CD}$, then by definition of $\overrightarrow{CD}$, we know $\underline{C\,f(B)} + \underline{f(B)D} = \underline{CD}$ or $\underline{CD} + \underline{Df(B)} = \underline{C\,f(B)}$

Case 1: $\underline{C\,f(B)} + \underline{f(B)D} = \underline{CD}$	Case 2: $\underline{CD} + \underline{Df(B)} = \underline{C\,f(B)}$
By substitution (of $\underline{CD} = \underline{Cf(B)}$)	By substitution (of $\underline{CD} = \underline{Cf(B)}$)
$\underline{CD} + \underline{f(B)D} = \underline{CD}$	$\underline{CD} + \underline{Df(B)} = \underline{CD}$
So	So
$\underline{f(B)D} = 0$	$\underline{f(B)D} = 0$
So by Ax 1c $f(B) = D$	So by Ax 1c $f(B) = D$

Therefore $f(B) = D$