Geometry Axioms and Theorems

Definition: The **plane** is a set of points that satisfy the axioms below. We will sometimes write E^2 to denote the plane.

Axiom 1: There is a metric on the points of the plane that is a distance function, which we will denote $d: E^2 \times E^2 \to [0,\infty)$. Given points $A, B \in E^2$, then $d(A,B)$ is called the **distance** between the points A and B, and we also use the notation: $d(A,B) = \underline{AB}$. The distance function satisfies the conditions

a) $d(A,B) = 0$ if and only if $A = B$	$\underline{AB} = 0$ if and only if $A = B$
b) $d(A,B) = d(B,A)$	$\underline{AB} = \underline{BA}$
c) If $A, B, C \in E^2$ then $d(A,B) + d(B,C) \geq d(A,C)$	$\underline{AC} \leq \underline{AB} + \underline{BC}$

Definition: Given two points $A, B \in E^2$, the **line segment** between them is defined to be the set: $\overline{AB} = \{X \in E^2 \mid \underline{AX} + \underline{XB} = \underline{AB}\}$

Definition: Given two points $A, B \in E^2$, the **ray** starting and the first, and going through the second is defined to be the set: $\overrightarrow{AB} = \{X \in E^2 \mid \underline{AX} + \underline{XB} = \underline{AB} \quad \text{or} \quad \underline{AB} + \underline{BX} = \underline{AX}\}$

Definition: Given two points $A, B \in E^2$, the **(infinite) line** between them is defined to be the set: $\overleftrightarrow{AB} = \{X \in E^2 \mid \underline{AX} + \underline{XB} = \underline{AB} \quad \text{or} \quad \underline{AB} + \underline{BX} = \underline{AX} \quad \text{or} \quad \underline{XA} + \underline{AB} = \underline{XB}\}$

Definition: Given a (center) point $A \in E^2$ and a (radius) distance $r \in [0,\infty)$, the **circle** with center A and radius r is defined to be $\{X \in E^2 \mid \underline{AX} = r\}$ and we will also denote $\odot(A,r)$

Definition: Four points $A, B, C, D \in E^2$ have **order** $A - B - C - D$ if $\underline{AB} + \underline{BC} + \underline{CD} = \underline{AD}$

Theorem 1: Given $A, B \in E^2$, then

a) $\overline{AB} = \overline{BA}$
b) $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$
c) $\overleftrightarrow{AB} = \overleftrightarrow{BA}$

Theorem 2: If points $A, B, C, D \in E^2$ have order $A - B - C - D$, then they have order $D - C - B - A$

Theorem 3: If points $A, B, C, D \in E^2$ have order $A - B - C - D$, then

a) $B \in \overline{AC}$
b) $B \in \overline{AD}$

Corollary 3.1: If $A, B, C, D \in E^2$ have order $A - B - C - D$, then

a) $C \in \overline{BD}$
b) $C \in \overline{AD}$

Definition: A set of points is **collinear** if there is an infinite line that contains all of the points.

Theorem 4: If points $A,B,C,D \in E^2$ have order $A-B-C-D$, then they are collinear.

hint: prove $B,C \in \overleftrightarrow{AD}$

Theorem 5: If $B \in \overleftrightarrow{AC}$ then $C \in \overleftrightarrow{AB}$

Hint: start by writing the definitions of $B \in \overleftrightarrow{AC}$ and $C \in \overleftrightarrow{AB}$

Corollary 5.1: If $B \in \overleftrightarrow{AC}$ then $A \in \overleftrightarrow{BC}$

Theorem 6: If four distinct points $W,X,Y,Z \in E^2$ have an order $W-X-Y-Z$, then each point lies on the line defined by any two of the other three points.

Axiom 2: If four points lie on the same line, then they have an order.

Theorem 7: If $C \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} \subseteq \overleftrightarrow{AC}$

Theorem 8: If $C \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

Theorem 9: If $C,D \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} = \overleftrightarrow{CD}$

Theorem 10: Distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in at most one point.

Definitions: A **function** $f: S \to T$ assigns each element in set *S* to one and only one element if set *T*, using the notation $f(s)$ where $s \in S$ and $f(s) \in T$ denotes the element that *s* is **mapped** to. A function is also called a **map** or a **mapping**. The set *S* is called the **domain** of the function. The set *T* is the **codomain** of the function. If every element of *T* is mapped to by some element of *S*, then *T* is also the **range** of the function.

A function is a **surjection**, which is an **onto** map or onto function if the codomain is the same as the range. In words, that means that every element of *T* is mapped to by some element of *S*. A set notation way of writing this is:

- A function $f: S \to T$ is onto if for every $t \in T$ then there exists an element $s \in S$ such that $f(s) = t$

In a typical proof that a function is onto, we choose a variable to represent an element in the codomain (let $t \in T$), then we do some algebra or similar work to represent an element in *S* in terms of *t* (then $g(t) \in S$) and use algebra or similar work show that the image of that element is *t* (then $f(g(t)) = t$).

A function is an **injection** which is a **one-to-one** map if no two elements in the domain are mapped to the same element in the codomain. Two set notation ways of writing this are:

- A function $f: S \to T$ is one-to-one if whenever $f(s) = f(x)$ then $s = x$
- A function $f: S \to T$ if whenever $s \neq x$ then $f(x) \neq f(s)$

In a typical proof that a function is 1-to-1, we do something similar to a proof by contradiction, which uses the first set notation definition. We assume that $x,s \in S$ and $f(x) = f(s)$. We then use algebra or similar reasoning to prove that $x = s$. We can then conclude that f is 1-to-1.

Definitions (continued) A function, $f: S \to T$, is invertible if there exists an inverse function $f^{-1}: T \to S$ such that $f \circ f^{-1}(t) = t$ for every $t \in T$ and $f^{-1} \circ f(s) = s$ for every $s \in S$.

Theorem 11: If $f: S \to T$ and $g: T \to U$ are functions such the domain of g is equal to the codomain of *f*, then $g \circ f: S \to U$ defined by $g \circ f(s) = g(f(s))$ is a function.

Theorem 12: If $f: S \to T$ and $g: T \to U$ are onto functions (surjections), then $g \circ f: S \to U$ defined by $g \circ f(s) = g(f(s))$ is an onto function (surjection).

Theorem 13: If $f: S \to T$ and $g: T \to U$ are one-to-one functions (injections), then $g \circ f: S \to U$ defined by $g \circ f(s) = g(f(s))$ is a one-to-one function (injection).

Theorem 14: A function $f: S \to T$ is one-to-one and onto (it is a bijection), if and only if it is invertible.

Definition: A one-to-one, onto function on the plane $f: E^2 \to E^2$ is called an **isometry** or a **rigid motion** if for every pair of points $X, Y \in E^2$, $d(X,Y) = d(f(X), f(Y))$. If S is any set in the plane, then $f(S)$ is its **isometric image.**

Axiom 3: For any non-collinear sets of points A, B, C and distinct points D, E and a specified side of $\overleftrightarrow{DE}$, there exists a unique isometry $f: E^2 \to E^2$ such that $f(A) = D$ and $f(B) \in \overrightarrow{DE}$ and $f(C)$ lies on the specified side of $\overleftrightarrow{DE}$. (Unique means there is one and only one such isometry)

Definition Given three non-collinear points A, B, C, the **triangle** ΔABC is the set $\overline{AB} \cup \overline{BC} \cup \overline{AC}$.

Definition: Two sets $S, T \in E^2$ are **congruent** if there is an isometry $f: E^2 \to E^2$ such that $f(S) = T$, where $f(S) = \{f(s) \mid s \in S\}$. For the specific cases of named objects, specifically segments, triangles, circles and polygons that are named by their vertices, saying that two of these are congruent also means that the isometry matches the named points in the order given. We will use the symbol $\cong$ to say that two sets or objects are congruent: $S \cong T$

Example: $\overline{AB} \cong \overline{CD}$ means that there is an isometry such that $f(A) = C$, $f(B) = D$ and $X \in \overline{AB}$ if and only if $f(X) \in \overline{CD}$

Theorem 15: The identity map is an isometry, and hence congruence is reflexive: If S is a subset of the plane, then $S \cong S$

Theorem 16: Every isometry has an inverse function that is also an isometry, and hence congruence is symmetric: If $S \cong T$ then $T \cong S$ (where S and T are subsets of the plane).

Theorem 17: The composition of two isometries is an isometry, and hence congruence is transitive: if $S \cong T$ and $T \cong V$ then $S \cong V$

Theorem 18: For an isometry $f: E^2 \to E^2$ and distinct points A, B and $f(A) = A'$ and $f(B) = B'$

- **a)** The isometric image of a line segment is a line segment: $f(\overline{AB}) = \overline{A'B'}$
- **b)** The isometric image of a ray is a ray $f(\overrightarrow{AB}) = \overrightarrow{A'B'}$
- **c)** The isometric image of a line is a line $f(\overleftrightarrow{AB}) = \overleftrightarrow{A'B'}$

Theorem 19: The isometric image of a circle is a circle, specifically, for an isometry $f: E^2 \to E^2$ and a circle $C = \odot(A, r)$ with center A and radius r (where A is a point and r is a positive real number), then $f(C)$ is the circle with center $f(A)$ and radius r. If we name $f(A) = A'$, then we can write the conclusion: $f(\odot(A, r)) = \odot(A', r)$

Theorem 20: Given segments $\overline{AB}$ and $\overline{CD}$ with equal lengths: $\underline{AB} = \underline{CD}$, there exists an isometry $f: E^2 \to E^2$ such that $f(A) = C$ and $f(B) = D$, and hence $\overline{AB} \cong \overline{CD}$.

Corollary 20.1: Two segments are congruent if and only if they have equal lengths (the distances between the endpoints are equal)

Theorem 21: Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, there exists an isometry $f: E^2 \to E^2$ such that $f(\overleftrightarrow{AB}) = \overleftrightarrow{CD}$ and hence $\overleftrightarrow{AB} \cong \overleftrightarrow{CD}$

Theorem 22: Given rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$, there exists an isometry $f: E^2 \to E^2$ such that $f(\overrightarrow{AB}) = \overrightarrow{CD}$ and hence $\overrightarrow{AB} \cong \overrightarrow{CD}$

Theorem 22.5: Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, side *S* of $\overleftrightarrow{AB}$, and isometry *f* such that $f(\overleftrightarrow{AB}) = \overleftrightarrow{CD}$, then $f(S)$ is a side of $\overleftrightarrow{CD}$.

Theorem 23: Given line $\overleftrightarrow{AB}$ and a point $C \notin \overleftrightarrow{AB}$, where the sets *S* and *T* are the two sides of $\overleftrightarrow{AB}$ that do and do not contain *C* respectively. Then there exists an isometry $f: E^2 \to E^2$ such that $f(\overleftrightarrow{AB}) = \overleftrightarrow{AB}$ and $f(S) = T$

Background: The classical (Euclid) definition of an angle only allows for angle measurements that are strictly between 0° and 180°. A trigonometry/calculus approach measures turning more than size and allows any real number (both positive and negative) to be a degree measurement. We will allow only positive angle measurements, and only degrees between 0 and 360 (inclusive). You can think of the angle having rays as edges and including the part of the plane that is between those edges. To avoid spending too much time on this, we will accept some things about angles without formally defining all of them. For example, we will not define what it means for a part of the plane to be between or be enclosed by two rays.

Definition: An **angle** is a subset of the plane that consists of two distinct rays (for example $\overrightarrow{AB}$ and $\overrightarrow{AC}$) that share a common origin point, and a section of the plane between the rays/enclosed by the rays. We write $\angle BAC$. If we need to specify which side of the plane is being enclosed by the rays, we will can write $\angle BAC, S$, where *S* is the subset of the plane that is enclosed by the rays.

Axiom 4: There is a function m that is an angle measurement, so that

- $m(\angle BAC) \in [0, 360°]$
- If $\angle ABC \cong \angle DEF$ then $m(\angle ABC) = m(\angle DEF)$
- Given $\angle BAC$, $\angle CAD$ and $\angle BAD$ such that *C* is in the section of the plane enclosed by $\angle BAD$, *B* is not in the section of the plane enclosed by $\angle CAD$ and *D* is not in the section of the plane enclosed by $\angle BAC$, then $m(\angle BAC) + m(\angle CAD) = m(\angle BAD)$
- The measure of the trivial angle that encloses nothing is 0°: $m(\angle BAB, \varnothing) = 0°$
- The measure of the trivial angle that encloses the whole plane is 360°: $m(\angle BAB, E^2) = 360°$

Theorem 25: The measure of a straight angle: $\angle BAC$ where $A \in \overleftrightarrow{BC}$ is 180°.

Theorem 26: If $d(A,B) = d(C,D)$ and there is an isometry, *f*, such that $f(A) = C$ and $f(B) \in \overrightarrow{CD}$ then $f(B) = D$

Theorem 27: If two angles have the same measure: $m(\angle ABC) = m(\angle DEF)$, and there is an isometry, *f*, such that $f(B) = E$, $f(A) \in \overrightarrow{ED}$, and $f(C)$ is on the same side of $\overleftrightarrow{DE}$ as the point F, then $f(C) \in \overrightarrow{EF}$.

Theorem 28: If two angles have the same measure, then they are congruent in both orders. I.e. If $m\angle ABC = m\angle DEF$, then there is an isometry, f, such that $f(B) = E$, $f(A) \in \overrightarrow{ED}$ and $f(C) \in \overrightarrow{EF}$ and there is another isometry g such that $g(B) = E$, $g(A) \in \overrightarrow{EF}$ and $g(C) \in \overrightarrow{ED}$.

Corollary 28.1 Two angles are congruent if and only if they have the same angle measure.

Theorem 29 (SAS): If a triangle has two sides and the included angle congruent to two sides and the included angle of another triangle, then the triangles are congruent

Note that in the case of triangle congruence, what you must prove is that there is an isometry that maps the vertices of one triangle onto the vertices of the other triangle.

Theorem 30 (ASA): If a triangle has two angles and the included side congruent to two angles and the included side of another triangle, then the triangles are congruent

Theorem 31 (Corresponding Sides of Congruent Triangles are Congruent: CPCTC): If two triangles are congruent, then their corresponding sides and angles are congruent.

Theorem 32 (Isosceles triangle theorem):

a) If two sides in a triangle are congruent, then the angles opposite the sides are also congruent.

b) If two angles in a triangle are congruent, then the sides opposite the angles are also congruent.

Theorem 33: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC} \cong \overline{AD}$ $\overline{BC} \cong \overline{BD}$ and $A \in \overline{CD}$, prove that ΔABC and ΔABD must satisfy the SAS property (two sides and the included angle of one triangle are congruent to two sides and the included angle of the triangle)

Theorem 34: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC} \cong \overline{AD}$ $\overline{BC} \cong \overline{BD}$ and $\overline{AB} \cap \overline{CD} = H$, where *H* is neither *A* nor *B*, prove that ΔABC and ΔABD must satisfy the SAS property (two sides and the included angle of one triangle are congruent to two sides and the included angle of the triangle)

Theorem 35: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC} \cong \overline{AD}$ $\overline{BC} \cong \overline{BD}$ and $\overleftrightarrow{AB} \cap \overline{CD} = H$, where *H* is not on segment $\overline{AB}$, prove that ΔABC and ΔABD must satisfy the SAS property (two sides and the included angle of one triangle are congruent to two sides and the included angle of the triangle)

Theorem 36: Given a pair of triangles with a shared side ΔABC and ΔABD where C and D are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC} \cong \overline{AD}$ $\overline{BC} \cong \overline{BD}$, then $\Delta ABC \cong \Delta ABD$

Theorem 37 (SSS): If a triangle has three sides congruent to the three sides of another triangle, then the triangles are congruent.

Hint: use an isometry to map one triangle closer to the other triangle

Theorem 38 (VA/vertical angles): If two lines intersect, then they make vertically opposite angles that are congruent to each other.

Definition: Given a line $\overleftrightarrow{AB}$, $C \in \overline{AB}$ and $D \notin \overleftrightarrow{AB}$, then the angles $\angle ACD$ and $\angle DCB$ (where each angle encloses a part of the plane that is a subset of a side of $\overleftrightarrow{AB}$) are called a ***linear pair*** because together they make up a line. (Note that an immediate result of this definition is that the angles are supplementary: $m(\angle ACD) + m(\angle DCB) = 180°$).

Definition: In a triangle (eg. ΔABC), the angles formed at the vertices of the triangle are called interior angles. If any side of the triangle is extended (eg. Extend $\overline{BC}$ to $\overline{BD}$ such that $C \in \overline{BD}$), then the angle formed between the triangle side and the extension of the intersection side (eg. $\angle ACD$, where the angle has measure less than $180°$) is called an ***exterior angle***. Note that an exterior angle always forms a linear pair with an adjacent interior angle.

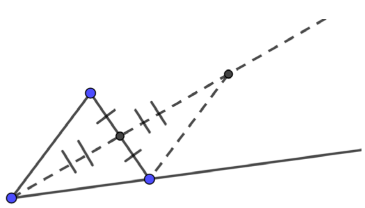
Axiom 5: If A and B are points, and r is a positive number, then there exists a point C such that $C \in \overrightarrow{AB}$ and $\underline{AC} = r$. Note that this is equivalent to saying that any circle and any ray whose vertex is the center of the circle intersect. This is related to saying that a circle and a ray are both continuous.

Theorem 39: Every segment has a midpoint.

Theorem 40: In any triangle, if one of the sides is extended, then the exterior angle is greater than either of the interior and opposite angles. In particular: if triangle ΔABC has side $\overline{BC}$ extended to $\overline{BD}$, then

a) The exterior angle has measure greater than the interior and opposite angle whose vertex is not on the extended side of the triangle.
b) The exterior angle has measure greater than the interior and opposite angle whose vertex is on the extended side of the triangle.

Hint for part a



Hint for part b: use part a and a previous theorem.

Theorem 41: In any triangle, the sum of any two of the three angles is less than $180°$

Theorem 42: Given triangle ΔABC and $D \in \overline{BC}$ then $\angle ADC > \angle ABC$

Definition: Two lines are ***parallel*** if they do not intersect. Two segments or rays are parallel if their associated infinite lines do not intersect.

Definition: If a line called a transversal (*t*) intersects two other lines (*l* and *m*) in two points ($t \cap l = A$ and $t \cap m = B$), then the ***interior angles*** are those angles that have the segment between the intersections as one of the sides of the angles, and all angle measures $< 180°$ (eg. $\angle CAB$ or $\angle DBA$ where $C \in l$ and $D \in m$)

Theorem 43: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and the interior angles on the same side of the transversal are supplementary (add to $180°$), then the interior angles on the other side of the transversal are also supplementary.

Theorem 44: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and the interior angles on the same side of the transversal are supplementary (sum is $180°$, then the lines (*l* and *m*) are parallel. `

> Hint: proof by contradiction—Note that the contradiction assumption creates a triangle. Use Theorem 40.

Definition: A ***right angle*** is an angle whose measure is $90°$. A ***right triangle*** is a triangle with one angle being a right angle.

Theorem 45: Given a line $\overleftrightarrow{AB}$, point $C \in \overline{AB}$ and point $D \notin \overleftrightarrow{AB}$ such that $\angle ACD \cong \angle DCB$, then the angles $\angle ACD$ and $\angle DCB$ are right angles.

Theorem 46: No triangle can have more than one right angle.

Theorem 47 (AAS): If two triangles have two angles and a non-included side congruent of one congruent to two angles and a corresponding non-included side of the other, then the triangles are congruent.

> Hint: use axiom 3 and theorem 40.

Definition: In a right triangle, the sides adjacent to the right angle are called the ***legs*** of the right triangle, and the side opposite the right angle is the ***hypotenuse***.

Theorem 48: If an angle has measure 180°, then it is a straight line.

Problem: Draw two non-congruent triangles that share SSA properties.

Theorem 49 (HL): If two right triangles have a leg and hypotenuse of one congruent to a leg and hypotenuse of the other, then the triangles are congruent.

> Hint: use axiom 3 to put the triangles back-to-back, with a shared leg.

Theorem 50: a. In any triangle with one side longer than another, the angle opposite the longer side is larger than the angle opposite the smaller side.
b. In any triangle with angle larger than another, the side opposite the larger angle is longer than the side opposite the smaller angle

Axiom 6 (Separation): An infinite line, a circle and a polygon all separate the plane into two sides, and if any segment, or arc includes a point on both sides of a separating figure (line, circle or polygon), then that arc and the separating figure have a point of intersection.

Theorem 51: Given a point and a line that does not include the point, there exists a line that includes that point, and is perpendicular to the given line.

Theorem 52: Given a line and a point not on the line, then the segment from the point to the line that is perpendicular to the line is:
a. unique (there is only one such perpendicular)
b. shortest (it is shorter than any other segment from the point to the line

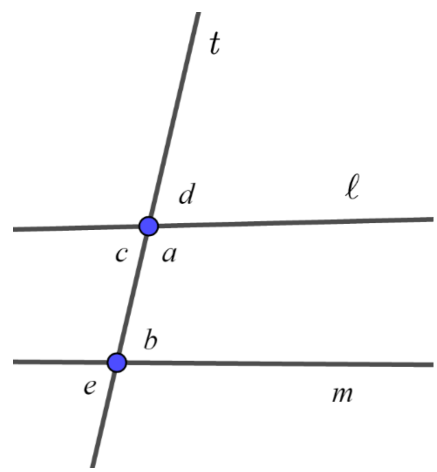
Axiom 7: Given two parallel lines, and a transversal that intersects them, then the sum of the interior angles on the same side of the transversal is $180°$.

Theorem 53: The following statements are equivalent

a) Axiom 7
b) Euclid's fifth postulate: If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.
c) Playfair's axiom: There is at most one line that can be drawn parallel to another given one through an external point.

Definition: Given a pair of lines (in the diagram ℓ, m), intersected at distinct points by a transversal (*t* in the diagram), then, using the angles named in the diagram:

- a and b are interior angles on the same side of the transversal
- b and c are alternate interior angles
- b and d are corresponding angles
- d and e are alternate exterior angles



Theorem 54: Given a pair of lines intersected at distinct points by a transversal, the following conditions are equivalent:

a) The angles in one pair of interior angles on the same side is supplementary
b) The angles in both pairs of interior angles on the same side are supplementary
c) The angles in one pair of alternate interior angles are congruent
d) The angles in both pairs of alternate interior angles are congruent
e) The angles in one pair of alternate exterior angles are congruent
f) The angles in both pairs of alternate exterior angles are congruent
g) The angles in one pair of corresponding angles are congruent
h) The angles in all pairs of corresponding angles are congruent.

Theorem 55: Given a line and a point not on the line, there exists a line parallel to the given line that includes the given point.

Theorem 56: Given a line and a point on the line, there exists a line perpendicular to the given line that includes the given point.

Theorem 57: The sum of the interior angles in a triangle is $180°$.

***Definition**:* A **quadrilateral** is a set of 4 vertices (A, B, C, D), and 4 segments, called **sides**, that connect the vertices in a circular order (AB, BC, CD and DA) such that no three vertices are collinear and no two segments intersect at a point other than an endpoint. Two vertices that are not connected by a side are called **opposite** vertices, and two vertices that are connected by a side are called **adjacent** vertices. Two sides that share a vertex are **adjacent**, and two sides that do not share a vertex are **opposite**. Segments that join opposite vertices are **diagonals** of the quadrilateral.

***Definition**:* A **parallelogram** is a quadrilateral with two pairs of parallel (opposite) sides.

***Definition**:* A **rhombus** is a quadrilateral with 4 congruent sides.

***Definition**:* A **rectangle** is a quadrilateral with 4 right angles

Theorem 58: Q1. The sum of interior angles of a quadrilateral is 360°

Theorem 59: P1. Every parallelogram has opposite congruent sides

Theorem 60: P2. Every parallelogram has opposite congruent angles

Theorem 61: P4. Every parallelogram has exterior angles that add up to 360°

Theorem 62: P5. The diagonals of a parallelogram intersect at the midpoint of the two diagonals

Theorem 63: P7. If a quadrilateral has opposite congruent angles then it is a parallelogram

Theorem 64: P6. If a quadrilateral has opposite congruent sides, then it is a parallelogram

Theorem 65: R2. Every rectangle has two set of parallel sides

Theorem 66: R1. Every rectangle has two sets of congruent sides

Theorem 67: R3. Every rectangle has congruent diagonals

Theorem 68: R6. If a quadrilateral has congruent opposite sides, and one 90° angle, then it is a rectangle

Theorem 69: Given a quadrilateral with one side designated as the base, if the sides adjacent to the base are congruent to each other and perpendicular to the base, then it is a rectangle. (hint: first show the diagonals are congruent)

Theorem 70: H4. The opposite angles in a rhombus are congruent

Theorem 71: H3. The adjacent angles of a rhombus are supplementary

Theorem 72: H2. Every rhombus is a parallelogram

Theorem 73: H6. The diagonals of a rhombus are perpendicular