Prove that the isometric image of a circle with radius 5 is a circle with radius 5.

Name a circle and an isometry
Let A be a point, and let $C = \text{cir}(A,5)$ be the circle with center A and radius 5.
Let $f : \text{Plane} \to \text{Plane}$ be an isometry.

Name the image of the point, and name the circle you are predicting will be the image
Let $f(A) = A'$ and let $S = \text{cir}(A',5)$ be the circle with center A' and radius 5.

Apply the definitions to the circles and the function images:
Then: $C = \{X \mid d(X,A) = 5\}$
$f(C) = \{f(X) \mid d(X,A) = 5\}$
$S = \{Y \mid d(Y,A') = 5\}$

<u>Part 1:</u>
Name a point in the first set
Let $Z \in f(C)$

Write down the properties you know from the definitions
Then there exists a point $X \in C$ such that $Z = f(X)$
And because $X \in C$, we know $d(X,A) = 5$

Write equations you know because of properties of f:
Because f preserves distances, $d(X,A) = d(Z,A')$

Substitute in the equations:
So, $d(Z,A') = 5$

Conclude that your point must also be in the second set, which means that the first set is a subset of the second.
Thus, $Z \in S$
Hence $f(C) \subseteq S$

<u>Part 2:</u>
Name a point in the second set
Let $W \in S$

Write down the properties you know from the definitions
So $d(W,A') = 5$

Write equations you know because of properties of f:
Because f is invertible, there exists a point V such that $f(V) = W$
Because f preserves distances, $d(V,A) = d(W,A')$

Substitute in the equations:
So, $d(V,A) = 5$

Conclude that your point must also be in the first set, which means that the second set is a subset of the first.
Thus, $V \in C$, and so $W = f(V) \in f(C)$
Hence $S \subseteq f(C)$

Write the final conclusion
Because $S \subseteq f(C)$ and $f(C) \subseteq S$, we can conclude that $f(C) = S$ and thus the image of a circle with radius 5 is a circle with radius 5.

Given that ΔABC is an equilateral triangle (all sides the same length), prove that the isometric image of A, B and C are vertices of an equilateral triangle.

In this problem, the points A, B, C are named already, but we still need to name the isometry and the images:
Let $f : \text{Plane} \rightarrow \text{Plane}$ be an isometry, and let $f(A) = A'$, $f(B) = B'$ and $f(C) = C'$

Write down the properties that we know from the givens/definitions
Since ΔABC is equilateral, we know that $d(A,B) = d(B,C) = d(A,C)$

Write down the properties that we know because f is an isometry:
Because f preserves distances, we know that $d(A,B) = d(A',B')$, $d(B,C) = d(B'C')$ and $d(A,C) = d(A'C')$

Substitute in the equations
Hence $d(A',B') = d(B',C') = d(A',C')$

Apply the definitions again
Therefore, the triangle $\Delta A'B'C'$ has equal length sides, and hence it is an equilateral triangle.

Notice that this proof doesn't use the step of showing that sets are the same because we are not looking at the isometric image of the whole triangle (a set with lots of points), we are just looking at the images of the three vertices. This doesn't prove that the image of the first triangle is the second triangle, it only proves that if the three points A, B, C are vertices of an equilateral triangle, the their images A'B'C' are also vertices of an equilateral triangle.