Plane Geometry

The **Plane** is a set of points that satisfy the axioms below

***Axiom 1*:** There is a distance function on the plane (also called a metric) $d : \text{Plane} \times \text{Plane} \to [0, \infty)$ with the properties:

- $d(A, B) = 0$ if and only if $A = B$
- $d(A, B) = d(B, A)$
- $d(A, C) \leq d(A, B) + d(B, C)$

where A, B, C are points in the plane. The distance between two points $d(A, B)$ can also be written AB

Defn: The *line segment* $\overline{AB}$ between A and B is the subset of the plane that contains all of the points of the plane that lie on the shortest path between A and B. In set notation: $\overline{AB} = \{X \mid d(A, X) + d(X, B) = d(A, B)\}$. The length of the line segment is defined to be the distance between the endpoints: $m(\overline{AB}) = d(A, B) = AB$.

The *ray* $\overrightarrow{AB}$ starting at A and passing through B consists of all of the points of the plane that lie on a shortest path with endpoint A that includes B, that is, $\overrightarrow{AB} = \{X \mid X \in \overline{AB} \text{ or } B \in \overline{AX}\}$

The *line* $\overleftrightarrow{AB}$ that includes A and B is the subset of the plane that contains all of the points of the plane that lie on a shortest path that includes both A and B, that is, $\{X \mid A \in \overline{BX} \text{ or } B \in \overline{AX} \text{ or } X \in \overline{AB}\}$

Problem 1: Rewrite the definitions of line and ray using distance equations (using the definition of a segment)

Defn: A *circle* with center point C and radius length $r > 0$ is the set of points in the plane that are distance r from C: $\text{cir}(C, r) = \{X \mid d(C, X) = r\}$

Defn: A set of points is *collinear* if all of the points lie on the same line.

Axiom 2: Every infinite line and every circle in the plane *separates* the plane into two *sides* (the sides are subsets of the plane that together make up all of the rest of the plane besides the separating shape) with the property that if the points A and B are on opposite sides of the shape, then any line segment or circle that contains both A and B also contains a point of the separating shape.

Defn. A function f is *invertible* if it has an inverse function f^{-1} such that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Axiom 3: There is a set of invertible distance-preserving mappings (functions) on the plane called *isometries* (also called isometric transformations, rigid transformations or rigid motions), such that for any non-collinear sets of points A, B, C and distinct points D, E and a specified side of $\overleftrightarrow{DE}$, there exists a unique isometry $f : \text{Plane} \to \text{Plane}$ such that $f(A) = D$ and $f(B) \in \overrightarrow{DE}$ and $f(C)$ lies on the specified side of $\overleftrightarrow{DE}$. (Unique means there is one and only one such isometry)

Problem 2: If you wanted an isometry that rotated the plane around point B through angle $\angle ABC$ (oriented from $\overrightarrow{BA}$ to $\overrightarrow{BC}$), how could you use the properties in axiom 3 to get one?

Problem 3: If you wanted an isometry that reflected the plane across line $\overleftrightarrow{AB}$, how could you use the properties in axiom 3 to get one?

Problem 4 (Challenge): If you wanted a rotation around point A by angle $\angle BCD$, how could you describe that by specifying 3 points?

Problem 5:

a) Is it allowed by axiom 3, given non-collinear points *A, B, C* and *D, E, F* to say that there is an isometry that sends A to F, B to a point on $\overrightarrow{FE}$ and C to the same side of $\overleftrightarrow{FE}$ as D? Why or why not?

b) Is it allowed by axiom 3, given non-collinear points *A, B, C* and *D, E, F* to say that there is an isometry that sends A to D, B to a point on $\overrightarrow{FE}$ and C to the same side of $\overleftrightarrow{FE}$ as D? Why or why not?

Defn: The *image* of a point, A under a function f is the point $f(A)$ that it is mapped to. The image of a set S under a function f is the set consisting of the images of all of the points in S, that is, $f(S) = \{f(X) \mid x \in S\}$. If the function is an isometry, then image of a set is called an *isometric image*.

Problem 6: Prove that if a function f is an isometry, then its inverse f^{-1} is also an isometry.

Theorem 1: Isometric images:

a) The isometric image of a line segment is a line segment
b) The isometric image of a ray is a ray
c) The isometric image of a line is a line
d) The isometric image of a circle is a circle with the same radius.

Axiom 4: If A and B are points, and r is a positive number, then there exists a point C such that $B \in \overline{AC}$ and $d(B,C) = r$

Defn: A set of four points *A, B, C*, and *D* is said to have the *order A-B-C-D* if
$d(A,B) + d(B,C) + d(C,D) = d(A,D)$

Theorem 2: Prove that points *A, B, C, D* have order *A-B-C-D* if and only if they have order *D-C-B-A*

Theorem 3: Prove that if four points have an order then they are collinear (there is a line that contains all four of them)

Axiom 5: If four points lie on the same line, then they have an order.

Theorem 4: If three points X, Y, Z satisfy a segment equation $d(X,Y) + d(Y,Z) = d(X,Z)$, then each point lies on the line defined by the other two. Ie. $X \in \overleftrightarrow{YZ}$, $Y \in \overleftrightarrow{XZ}$ and $Z \in \overleftrightarrow{XY}$

Theorem 5: If four distinct points have an order $W - X - Y - Z$, then each point lies on the line defined by any two of the other three points.

Theorem 6: If $C \in \overrightarrow{AB}$ then $\overrightarrow{AB} \subseteq \overleftrightarrow{AC}$

Theorem 7: If $B \in \overrightarrow{AC}$ then $C \in \overrightarrow{AB}$

Theorem 8: If $C \in \overrightarrow{AB}$ then $\overrightarrow{AB} = \overrightarrow{AC}$

Theorem 9: If $C, D \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} = \overleftrightarrow{CD}$

Theorem 10: Distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in at most one point.

Axiom 6: Two distinct rays that share a common origin point separate the plane into two sides.

Defn, An *angle* consists of two distinct rays (for example $\overrightarrow{AB}$ and $\overrightarrow{AC}$) that share a common origin point. We write $\angle BAC$. If we wish to consider a side of the angle together with the rays, then we can allow for angle measurements that are greater than $180°$. In this document, we will use the notation $\measuredangle BAC$ or occasionally $\measuredangle BAC, side$ when we are specifying a ray and a side, and we will call that a *solid angle*. $\measuredangle BAC$ without a side denotes the solid angle where the side does not include $\overleftrightarrow{AB}$.

Defn. Two subsets of the plane (eg, segments, circles or angles) are *congruent* if there is an isometry that maps one onto the other. The symbol $\cong$ denotes congruence.

Theorem 11: A segment is congruent to itself in the opposite order. Ie. $\overline{AB} \cong \overline{BA}$

Theorem 12: An angle is congruent to itself in the opposite order. Ie. $\angle BAC \cong \angle CAB$

Theorem 13: The identity map is an isometry, and hence congruence is reflexive: If S is a subset of the plane, then $S \cong S$

Theorem 14: The inverse of an isometry is also an isometry, and hence congruence is symmetric: If $S \cong T$ then $T \cong S$ (where S and T are subsets of the plane).

Theorem 15: The composition of two isometries is an isometry, and hence congruence is transitive: if $S \cong T$ and $T \cong V$ then $S \cong V$

Theorem 16: If two segments are congruent, then they have the same length.

Axiom 7: No segment, circle or polygon is congruent to a proper subset of itself. No angle is congruent to a proper subset of itself, and no solid angle is congruent to a proper subset of itself in a way that preserves the origin point.

Axiom 8: There is a function m that is an angle measurement, so that

- $m(\angle BAC) \in (0,180°]$ or $m(\measuredangle ABC, side) \in (0,360°)$
- If $\angle ABC \cong \angle DEF$ then $m(\angle ABC) = m(\angle DEF)$
- $m(\angle BAC) = m(\measuredangle BAC, side)$ if $\measuredangle BAC, side$ does not include $\overleftrightarrow{AB}$ (otherwise $360° - m(\angle BAC) = m(\measuredangle BAC, side)$)
- If $\measuredangle BAC, s1 \cup \measuredangle CAD, s2 = \measuredangle BAD, s3$ then $m(\measuredangle BAC, s1) + m(\measuredangle CAD, s2) = m(\measuredangle BAD, s3)$