Plane Geometry

The **Plane** is a set of points that satisfy the axioms below

***Axiom 1*:** There is a distance function on the plane (also called a metric) $d:\text{Plane}\times\text{Plane}\to[0,\infty)$ with the properties:

- $d(A,B)=0$ if and only if $A=B$
- $d(A,B)=d(B,A)$
- $d(A,C)\le d(A,B)+d(B,C)$

where A,B,C are points in the plane. The distance between two points $d(A,B)$ can also be written AB

Defn: The *line segment* $\overline{AB}$ between A and B is the subset of the plane that contains all of the points of the plane that lie on the shortest path between A and B. In set notation: $\overline{AB}=\{X\mid d(A,X)+d(X,B)=d(A,B)\}$. The length of the line segment is defined to be the distance between the endpoints: $m(\overline{AB})=d(A,B)=AB$.

The *ray* $\overrightarrow{AB}$ starting at A and passing through B consists of all of the points of the plane that lie on a shortest path with endpoint A that includes B, that is, $\overrightarrow{AB}=\{X\mid X\in\overline{AB}\text{ or }B\in\overline{AX}\}$

The *line* $\overleftrightarrow{AB}$ that includes A and B is the subset of the plane that contains all of the points of the plane that lie on a shortest path that includes both A and B, that is, $\{X\mid A\in\overline{BX}\text{ or }B\in\overline{AX}\text{ or }X\in\overline{AB}\}$

Problem 1: Rewrite the definitions of line and ray using distance equations (using the definition of a segment)

Defn: A *circle* with center point C and radius length $r>0$ is the set of points in the plane that are distance r from C: $\text{cir}(C,r)=\{X\mid d(C,X)=r\}$

Defn: A set of points is *collinear* if all of the points lie on the same line.

Axiom 2: Every infinite line and every circle in the plane *separates* the plane into two *sides* (the sides are subsets of the plane that together make up all of the rest of the plane besides the separating shape) with the property that if the points A and B are on opposite sides of the shape, then any line segment or circle that contains both A and B also contains a point of the separating shape.

Defn. A function f is *invertible* if it has an inverse function f^{-1} such that $f(f^{-1}(x))=x$ and $f^{-1}(f(x))=x$

Axiom 3: There is a set of invertible distance-preserving mappings (functions) on the plane called *isometries* (also called isometric transformations, rigid transformations or rigid motions), such that for any non-collinear sets of points A,B,C and distinct points D,E and a specified side of $\overleftrightarrow{DE}$, there exists a unique isometry $f:\text{Plane}\to\text{Plane}$ such that $f(A)=D$ and $f(B)\in\overrightarrow{DE}$ and $f(C)$ lies on the specified side of $\overleftrightarrow{DE}$. (Unique means there is one and only one such isometry)

Problem 2: If you wanted an isometry that rotated the plane around point B through angle $\angle ABC$ (oriented from $\overrightarrow{BA}$ to $\overrightarrow{BC}$), how could you use the properties in axiom 3 to get one?

Problem 3: If you wanted an isometry that reflected the plane across line $\overleftrightarrow{AB}$, how could you use the properties in axiom 3 to get one?

Problem 4 (Challenge): If you wanted a rotation around point A by angle $\angle BCD$, how could you describe that by specifying 3 points?

Problem 5:

a) Is it allowed by axiom 3, given non-collinear points *A, B, C* and *D, E, F* to say that there is an isometry that sends A to F, B to a point on $\overrightarrow{FE}$ and C to the same side of $\overleftrightarrow{FE}$ as D? Why or why not?

b) Is it allowed by axiom 3, given non-collinear points *A, B, C* and *D, E, F* to say that there is an isometry that sends A to D, B to a point on $\overrightarrow{FE}$ and C to the same side of $\overleftrightarrow{FE}$ as D? Why or why not?

Defn: The *image* of a point, A under a function f is the point $f(A)$ that it is mapped to. The image of a set S under a function f is the set consisting of the images of all of the points in S, that is, $f(S) = \{f(X) \mid x \in S\}$. If the function is an isometry, then image of a set is called an *isometric image*.

Problem 6: Prove that if a function f is an isometry, then its inverse f^{-1} is also an isometry.

Theorem 1: Isometric images:

a) The isometric image of a line segment is a line segment
b) The isometric image of a ray is a ray
c) The isometric image of a line is a line
d) The isometric image of a circle is a circle with the same radius.

Axiom 4: If A and B are points, and r is a positive number, then there exists a point C such that $B \in \overline{AC}$ and $d(B,C) = r$

Defn: A set of four points *A, B, C,* and *D* is said to have the *order A-B-C-D* if
$d(A,B) + d(B,C) + d(C,D) = d(A,D)$

Theorem 2: Prove that points *A, B, C, D* have order *A-B-C-D* if and only if they have order *D-C-B-A*

Theorem 3: Prove that if four points have an order then they are collinear (there is a line that contains all four of them)

Axiom 5: If four points lie on the same line, then they have an order.

Theorem 4: If three points X, Y, Z satisfy a segment equation $d(X,Y) + d(Y,Z) = d(X,Z)$, then each point lies on the line defined by the other two. Ie. $X \in \overleftrightarrow{YZ}$, $Y \in \overleftrightarrow{XZ}$ and $Z \in \overleftrightarrow{XY}$

Theorem 5: If four distinct points have an order $W - X - Y - Z$, then each point lies on the line defined by any two of the other three points.

Theorem 6: If $C \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} \subseteq \overleftrightarrow{AC}$

Theorem 7: If $B \in \overleftrightarrow{AC}$ then $C \in \overleftrightarrow{AB}$

Theorem 8: If $C \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} = \overleftrightarrow{AC}$

Theorem 9: If $C, D \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AB} = \overleftrightarrow{CD}$

Theorem 10: Distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in at most one point.

Axiom 6: Two distinct rays that share a common origin point separate the plane into two sides.

Defn, An *angle* consists of two distinct rays (for example $\overrightarrow{AB}$ and $\overrightarrow{AC}$) that share a common origin point. We write $\angle BAC$. If we wish to consider a side of the angle together with the rays, then we can allow for angle measurements that are greater than $180°$. In this document, we will use the notation $\measuredangle BAC$ or occasionally $\measuredangle BAC, side$ when we are specifying a ray and a side, and we will call that a *solid angle*. $\measuredangle BAC$ without a side denotes the solid angle where the side does not include $\overleftrightarrow{AB}$.

Defn. Two subsets of the plane (eg, segments, circles or angles) are *congruent* if there is an isometry that maps one onto the other. The symbol $\cong$ denotes congruence.

Theorem 11: A segment is congruent to itself in the opposite order. Ie. $\overline{AB} \cong \overline{BA}$

Axiom 7: No segment, circle or polygon is congruent to a proper subset of itself. No angle is congruent to a proper subset of itself, and no solid angle is congruent to a proper subset of itself in a way that preserves the origin point.

Theorem 12: The identity map is an isometry, and hence congruence is reflexive: If S is a subset of the plane, then $S \cong S$

Theorem 13: The inverse of an isometry is also an isometry, and hence congruence is symmetric: If $S \cong T$ then $T \cong S$ (where S and T are subsets of the plane).

Theorem 14: The composition of two isometries is an isometry, and hence congruence is transitive: if $S \cong T$ and $T \cong V$ then $S \cong V$

Theorem 15: An angle is congruent to itself in the opposite order. Ie. $\measuredangle BAC \cong \measuredangle CAB$

Theorem 16: If two segments are congruent, then they have the same length.

Axiom 8: There is a function m that is an angle measurement, so that

- $m(\angle BAC) \in (0,180°]$ or $m(\measuredangle ABC, side) \in (0,360°)$
- If $\angle ABC \cong \angle DEF$ then $m(\angle ABC) = m(\angle DEF)$
- $m(\angle BAC) = m(\measuredangle BAC, side)$ if $\measuredangle BAC, side$ does not include $\overleftrightarrow{AB}$ (otherwise $360° - m(\angle BAC) = m(\measuredangle BAC, side)$)
- If $\measuredangle BAC, s1 \cup \measuredangle CAD, s2 = \measuredangle BAD, s3$ then $m(\measuredangle BAC, s1) + m(\measuredangle CAD, s2) = m(\measuredangle BAD, s3)$
- $m(\angle ABC) = 180°$ if and only if $B \in \overline{AC}$

Theorem 17: If two segments have the same length then they are congruent.

Theorem 17a: If $d(A,B)=d(C,D)$ and there is an isometry, *f*, such that $f(A)=C$ and $f(B)\in\overrightarrow{CD}$ then $f(B)=D$

Theorem 18: If two angles have the same measure, then they are congruent in both orders. I.e. If $m\measuredangle ABC = m\measuredangle DEF$, then there is an isometry, f, such that $f(B)=E$, $f(A)\in\overrightarrow{ED}$ and $f(C)\in\overrightarrow{EF}$ and there is another isometry g such that $g(B)=E$, $g(A)\in\overrightarrow{EF}$ and $g(C)\in\overrightarrow{ED}$.

Theorem 18a: If two angles have the same measure: $m(\measuredangle ABC)=m(\measuredangle DEF)$, and there is an isometry, *f*, such that $f(B)=E$, $f(A)\in\overrightarrow{ED}$, and $f(C)$ is on the same side of $\overleftrightarrow{DE}$ as the point F, then $f(C)\in\overrightarrow{EF}$.

Theorem 19 (SAS): If a triangle has two sides and the included angle congruent to two sides and the included angle of another triangle, then the triangles are congruent

Note that in the case of triangle congruence, what you must prove is that there is an isometry that maps the vertices of one triangle onto the vertices of the other triangle.

Theorem 20 (ASA): If a triangle has two angles and the included side congruent to two angles and the included side of another triangle, then the triangles are congruent

Theorem 21 (Isosceles triangle theorem):

Part 1: if two sides in a triangle are congruent, then the angles opposite the sides are also congruent.
Part 2: if two angles in a triangle are congruent, then the sides opposite the angles are also congruent.

Theorem 22: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC}\cong\overline{AD}$ $\overline{BC}\cong\overline{BD}$ and $A\in\overline{CD}$, prove that ΔABC and ΔABD must satisfy the SAS property (two sides and the included angle of one triangle are congruent to two sides and the included angle of the triangle)

Theorem 23: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC}\cong\overline{AD}$ $\overline{BC}\cong\overline{BD}$ and $\overline{AB}\cap\overline{CD}=H$, where *H* is neither *A* nor *B*, prove that ΔABC and ΔABD must satisfy the SAS property (two sides and the included angle of one triangle are congruent to two sides and the included angle of the triangle)

Theorem 24: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC}\cong\overline{AD}$ $\overline{BC}\cong\overline{BD}$ and $\overleftrightarrow{AB}\cap\overline{CD}=H$, where *H* is not on segment $\overline{AB}$, prove that ΔABC and ΔABD must satisfy the SAS property (two sides and the included angle of one triangle are congruent to two sides and the included angle of the triangle)

Theorem 25: Given a pair of triangles with a shared side ΔABC and ΔABD where *C* and *D* are on opposite sides of line $\overleftrightarrow{AB}$ such that $\overline{AC}\cong\overline{AD}$ $\overline{BC}\cong\overline{BD}$, then $\Delta ABC\cong\Delta ABD$

Theorem 26 (SSS): If a triangle has three sides congruent to the three sides of another triangle, then the triangles are congruent.

Theorem 28 (VA/vertical angles): If two lines intersect, then they make vertically opposite angles that are congruent to each other.

Theorem 29 (lemma to theorem 30): Given triangle ΔABC and segment $\overline{BD}$ such that $C \in \overline{BD}$, midpoint $E \in \overline{AC}$ such that $\overline{AE} \cong \overline{EC}$ and point F such that $E \in \overline{BF}$ and $\overline{BE} \cong \overline{EF}$, then $\angle EAB \cong \angle ECF$

Definition: Given a line $\overleftrightarrow{AB}$, $C \in \overline{AB}$ and $D \notin \overleftrightarrow{AB}$, then the angles $\angle ACD$ and $\angle DCB$ are called a ***linear pair*** because together they make up a line. (Note that an immediate result of this definition is that the angles are supplementary: $m(\angle ACD) + m(\angle DCB) = 180°$).

Definition: In a triangle (eg. ΔABC), the angles formed at the vertices of the triangle are called interior angles. If any side of the triangle is extended (eg. Extend $\overline{BC}$ to $\overline{BD}$ such that $C \in \overline{BD}$), then the angle formed between the triangle side and the extension of the intersection side (eg. $\angle ACD$) is called an ***exterior angle***. Note that an exterior angle always forms a linear pair with an adjacent interior angle.

Theorem 30: In any triangle, if one of the sides is extended, then the exterior angle is greater than either of the interior and opposite angles.

Theorem 31: In any triangle, the sum of any two of the three angles is less than $180°$

Theorem 32: Given triangle ΔABC and $D \in \overline{BC}$ then $\angle ADC > \angle ABC$

Theorem 33 (AAS): If two triangles have two angles and a non-included side congruent of one congruent to two angles and a corresponding non-included side of the other, then the triangles are congruent.

Definition: A ***right angle*** is an angle whose measure is $90°$. A ***right triangle*** is a triangle with one angle being a right angle.

Theorem 34: Given a line $\overleftrightarrow{AB}$, point $C \in \overline{AB}$ and point $D \notin \overleftrightarrow{AB}$ such that $\angle ACD \cong \angle DCB$, then the angles $\angle ACD$ and $\angle DCB$ are right angles.

Theorem 35: No triangle can have more than one right angle.

Definition: In a right triangle, the sides adjacent to the right angle are called the ***legs*** of the right triangle, and the side opposite the right angle is the ***hypotenuse***.

Theorem 36 (HL): If two right triangles have a leg and hypotenuse of one congruent to a leg and hypotenuse of the other, then the triangles are congruent.

Definition: Two lines are ***parallel*** if they do not intersect. Two segments or rays are parallel if their associated infinite lines do not intersect.

Definition: If a line called a transversal (*t*) intersects two other lines (*l* and *m*) in two points ($t \cap l = A$ and $t \cap m = B$), then the ***interior angles*** are those angles that have the segment between the intersections as one of the sides of the angles (eg. $\angle CAB$ or $\angle DBA$ where $C \in l$ and $D \in m$)

Theorem 37: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and the interior angles on the same side of the transversal are supplementary (add to $180°$), then the interior angles on the other side of the transversal are also supplementary.

Theorem 38: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and the interior angles on the same side of the transversal are supplementary, then the lines (*l* and *m*) are parallel.

Axiom 9 (Euclid's Parallel postulate): If a transversal (*t*) intersects each of a pair of lines (*l* and *m*) in two distinct points, and the sum of the measures of the interior angles on one side of the transversal is strictly less than $180°$, then the lines (*l* and *m*) intersect at a point that is on the same side of the transversal as those interior angles.

Theorem 39: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and the sum of the measures of the interior angles on one side of the transversal is strictly greater than $180°$, then the sum of the measures of the interior angles on the other side of the transversal is strictly less than $180°$.

Theorem 40: If a transversal (*t*) intersects each of a pair of parallel lines (*l* and *m*), then the interior angles on the same side of the transversal are supplementary (add to $180°$).

Hint: proof by contradiction

Defn: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points (*A* and *B*), then a pair of interior angles are called alternate interior angles if the interior angle at *A* is on the opposite side of *t* from the interior angle at *B*.

Theorem 41: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and if the angles in one pair of alternate interior angles are congruent, then the angles in the other pair of alternate interior angles are congruent.

Theorem 42 (AIA Parallel): If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, then

a. If the angles in a pair of alternate interior angles are congruent, then the lines (*l* and *m*) are parallel
b. If the lines (*l* and *m*) are parallel, then the angles in both pairs of alternate interior angles are congruent.

Defn: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points (*A* and *B*), then a pair of corresponding angles consists of an interior angle at one of the intersection points ($\angle ABC$ or $\angle BAD$), and an angle at the other intersection point ($\angle EAF$ or $\angle GBH$) that is not an interior angle, and that is on the same side of the transversal *t* as the first angle in the pair. (For example, $\angle ABC$ and $\angle EAF$, where $E \in t$ and *E* is on the opposite side of *l* from *B*, and $F \in l$ is on the same side of *T* as *C*.

Theorem 43: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and if the angles in one pair of corresponding angles are congruent, then the angles in the other three pairs of corresponding angles are congruent.

Theorem 44 (CA Parallel): If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, then

a. If the angles in a pair of corresponding angles are congruent, then the lines (*l* and *m*) are parallel
b. If the lines (*l* and *m*) are parallel, then the angles in all pairs of corresponding angles are congruent.

Theorem 45: If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, and if the angles in one pair of alternate exterior angles are congruent, then the angles in the other pair of alternate exterior angles are congruent.

Theorem 46 (AEA Parallel): If a transversal (*t*) intersects two other lines (*l* and *m*) in two distinct points, then

a. If the angles in a pair of alternate exterior angles are congruent, then the lines (*l* and *m*) are parallel
b. If the lines (*l* and *m*) are parallel, then the angles in both pairs of alternate exterior angles are congruent.

Theorem 47 (Create parallels) Given any line, and a point that is not on the line, there is another line that is parallel to the first line, and which contains the given point.

Theorem 48: The interior angles of a triangle add up to 180º.

Definition: A **quadrilateral** is a set of 4 vertices (A, B, C, D), and 4 segments, called **sides**, that connect the vertices in a circular order (AB, BC, CD and DA) such that no three vertices are collinear and no two segments intersect at a point other than an endpoint. Two vertices that are not connected by a side are called **opposite** vertices, and two vertices that are connected by a side are called **adjacent** vertices. Two sides that share a vertex are **adjacent**, and two sides that do not share a vertex are **opposite**. Segments that join opposite vertices are **diagonals** of the quadrilateral.

Definition: A **parallelogram** is a quadrilateral with two pairs of parallel (opposite) sides.
Definition: A **rhombus** is a quadrilateral with 4 congruent sides.
Definition: A **rectangle** is a quadrilateral with 4 right angles
Definition: A **kite** is a quadrilateral with 2 (disjoint) congruent adjacent sides.
Definition: Two lines are perpendicular if they meet at a right angle.

Theorem 49: The angles in a quadrilateral add up to 360°

Theorem 50: The adjacent angles of a parallelogram are supplementary

Theorem 51: The opposite angles of a parallelogram are congruent

Theorem 52: The opposite sides of a parallelogram are congruent.

Theorem 53: If the opposite sides of a quadrilateral are congruent, then it is a parallelogram

Theorem 54: If opposite angles in a quadrilateral are supplementary to the same adjacent angle, then it is a parallelogram

Theorem 55: The opposite sides of a rhombus are parallel.

Theorem 56: If the diagonals of a quadrilateral bisect each other and are perpendicular, then it is a rhombus

Theorem 57: Every rectangle is a parallelogram.

Theorem 58: The opposite sides of a rectangle are congruent.

Theorem 59: (Euclid). Given a quadrilateral with one side called a base, and the sides adjacent to the base are called the legs, if the legs congruent to each other, and are perpendicular to the base, then it is a rectangle.

Theorem 60: The angles at the vertices of the non-adjacent sides of a kite are congruent.

Theorem 61: The angle at the vertex of the shorter pair of congruent sides is larger than the angle at the vertex of the longer pair of congruent sides.

Theorem 62: Given a line, and a point on the line, there exists a line that is perpendicular to the first line, and passes through the given point.

Definition: A square is a rectangle that is also a rhombus.

Theorem 63: Given a segment, there exists a square where one side of the square is the given segment.

Axiom 10 (Areas): Any finite part of the plane that is bounded by line segments (interior of a polygon) has a number associated with it called the ***area***. We will sometimes write $\text{area}(\Delta ABC)$ or $\text{area}(ABCD)$ to denote the area of a polygon. The area of a polygon satisfies the following properties:

- If the length of a side of a square is 1, then the area of the square is 1.
- If two polygons are congruent, then their areas are equal
- If a polygon is cut into two polygons, then the area of the larger polygon is the sum of the areas of the two smaller polygons.

Theorem 64: If the lengths of two adjacent sides of a rectangle are b and h, then its area is $b\cdot h$

Theorem 65: If the lengths of the legs of a right triangle are b and h, then the area of the triangle is $\frac{1}{2}bh$

Definition: In a parallelogram, any side of the parallelogram can be called the ***base***. Once a base is chosen, then the ***height*** of the parallelogram is a segment (or the length of that segment) that is perpendicular to the base, and where one endpoint is on the base, and the other endpoint is on the line containing the side opposite to the base.

Theorem 66: Given a line and a point not on the line, there exists a line that is perpendicular to the given line, and that passes through the given point.

Theorem 67: If a base and height of a parallelogram are specified, and their lengths are b and h, then the area of the parallelogram is $b\cdot h$

Definition: In a triangle, any side of the triangle can be called the ***base***. Once a base is chosen, then the ***height*** of the triangle is the segment that is perpendicular to the base, and where one endpoint is the vertex opposite the base, and the other endpoint is on base.

Theorem 68: In a triangle if a base is specified, and the base and height lengths are b and h, then the area of the triangle is $\frac{1}{2}bh$

Theorem 68 (Pythagorean Theorem): Given a right triangle with legs of length a and b and hypotenuse of length c, then the area of a square constructed on the hypotenuse is equal to the sum of the areas of the squares constructed on the two legs.

Theorem 69 (Similarity Lemma): Given a triangle ΔABC with points $D \in \overline{AB}$ and $E \in \overline{AC}$ then $\overline{BC}$ is parallel to $\overline{DE}$ if and only if the side lengths are proportional: $\frac{AD}{AB} = \frac{AE}{AC}$ where $AD = d(A,D)$, etc.

Theorem 70 (AAA Similarity): Given triangles ΔABC and ΔDEF that have congruent corresponding angles, then their side lengths are proportional: $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BD}{EF}$