Theorem 9.

Given $\triangle ABC$ and $\triangle DEF$

Make circle 1 with center A and radius AD $d(A,D)$

and circle 2 w/ center D and radius AD

Let P be one of the points of intersection.

Let f be the rotation around P by $\angle APD$

Let $f(A)=A'$, $f(B)=B'$, $f(C)=C'$

So $A'=D$

To prove: $f(A)=A'=D$

by defn. of rotation,

$\angle APA' \cong \angle APD$

and $\angle APA'$ and $\angle APD$ share a side

(and A' & D are on same side of $\overleftrightarrow{AP}$)

by Thm 1 $\overrightarrow{PA'} = \overrightarrow{PD}$ (1)

the same ray

$d(A,P) = d(A,D)$

radius of circle 1

$d(D,P) = d(A,D)$

radius of circle 2.

$d(A',P) = d(A,P)$ because f is an isometry

$\underset{\parallel}{}$ $d(f(A),f(P))$

so $d(D,P) = d(A',P)$ (2)

same pt

Thus $A'=D$ (1)(2)

by theorem 2

Let g be the rotation around D by $\angle B'DE$

Let $g(A') = A''$
$g(B') = B''$
$g(C') = C''$

so $B'' \in \overrightarrow{DE}$

and $A'' = D$ (3)
because $A' = D$ is a fixed point

if C'' is not on the same side of $\overleftrightarrow{DE}$ as F then

let h be the reflection across $\overleftrightarrow{DE}$

and Let $h(A'') = A'''$
$h(B'') = B'''$
$h(C'') = C'''$

To prove: $B'' \in \overrightarrow{DE}$

by defn. of rotation,
$\angle B'DB'' \cong \angle B'DE$
and $\angle B'DB''$ and $\angle B'DE$ share a side
(and B'' & E are on the same side of $\overleftrightarrow{B'D}$)

By thm 1 $\overrightarrow{DB''} = \overrightarrow{DE}$
so $B'' \in \overrightarrow{DE}$ (4)

Case 1: C'' is on the same side of $\overleftrightarrow{DE}$ as F (5)
then $g \circ f(A) = D$ (3)
and $g \circ f(B) \in \overrightarrow{DE}$ (4)
and $g \circ f(C)$ is on same side of $\overleftrightarrow{DE}$ as F (5)
and $g \circ f$ is an isometry
by thm 8

Case 2: C'' is not on the same side of $\overleftrightarrow{DE}$ as F
then $h \circ g \circ f(A) = D$ $\quad D \in \overleftrightarrow{DE}$ so D is fixed
and $h \circ g \circ f(B) \in \overrightarrow{DE}$ $\quad \overrightarrow{DE} \subset \overleftrightarrow{DE}$ so B'' is fixed
and $h \circ g \circ f(C)$ on same side of $\overleftrightarrow{DE}$ as F because reflections switch sides
and $h \circ g \circ f$ is an isometry
by thm 8