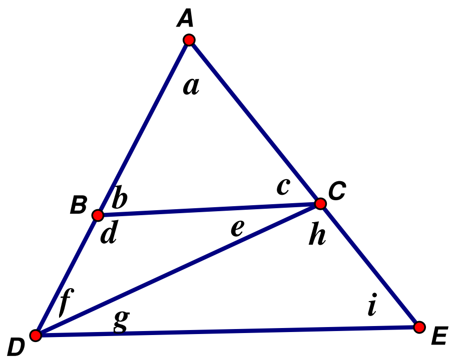
Proving that the sum of angles in ΔABC is smaller than the sum of angles in ΔADE given that these are triangles on a sphere.

Proof: Let the measures of the angles be named as in the diagram below.

We wish to prove that a + b + c < a + f + g + i



d+b=180° (1)

c+e+h=180° (2) because the angles make up a straight line

a+b+c>180° (3)

d+e+f>180° (4)

g+h+i>180° (5) because these are triangles on a sphere.

From here there are several different strategies that work. Stuff in (**) is my commentary, the rest of it is the proof. Look at the 3 strategies and pick one that you like:

Strategy 1: (*we want to get a+f+g+i to be bigger than something. Lets solve some inequalities to get those measurements bigger than something (especially f, g, and i: a is the same on both sides of the desired inequality:*)

d+e+f > 180 so f > 180-d-e (4)

g+h+i > 180 so g+i > 180-h (5)

If we add those together, we get:

f+g+i >360-d-e-h

(*Now I want to get rid of d, e, and h, but they're negative, so really I want to solve for -d, -e and -h. I haven't used the first equations yet—lets try those*)

d+b = 180 so d = 180-b and -d = b-180 (1)

c+e+h = 180 so e+h = 180-c and -e-h = c-180 (2)

Substituting those into the right side of

f+g+i > 360-d-e-h

we get

f+g+i > 360+(b-180)+(c-180)

so after simplifying, we get:

f+g+i > b+c

and adding a to both sides give us

a+f+g+i > a+b+c which is what we were trying to prove.

This way feels like its relying a little bit on luck to make everything come together, but with only 5 equations it works out pretty fast

Strategy 2: (*pick one side of the thing you're trying to prove, and use the other equations to relate it to the other side. I'm going to start with a+f+g+i because it has more terms)

$d+e+f > 180$ so $f > 180-d-e$ (3)

Thus $a+f+g+i > a+(180-d-e)+g+i$

and $g+h+i > 180$ so $g+i > 180-h$ (4)

Thus

$a+(180-d-e)+g+i > a+(180-d-e)+(180-h)$

and

$a+(180-d-e)+(180-h) = a+360-d-e-h = a+360 + -d + (-e-h)$

so $a+f+g+i > a+360 + -d + -e-h$

Now

$d+b = 180$ so $d = 180-b$ and $-d = b-180$ (1)

$c+e+h = 180$ so $e+h = 180-c$ and $-e-h=c-180$ (2)

And by substitution:

$a+360 -d -e-h = a+360+(b-180)+(c-180) = a+b+c$

Putting it all together:

$a+f+g+i > a+(180-d-e)+g+i > a+(180-d-e)+(180-h) = a+360+-d +-e-h = a+360+(b-180)+(c-180) = a+b+c$

Thus

$a+f+g+i > a+b+c$

This way the main thing to watch out for is that you never have an equation where you're doing the same thing to both sides—you're making a big long chain of equations and inequalities where each one works because of one of the given equations and inequalities.

Strategy 3: (*Look at all of the angles in the larger triangle, and put an equation together from that which is the same as the sum of angles in the large triangle:

$a+f+g+i = a+(b+d-180)+f+g+i+(e+c+h-180)$

(*Notice that this uses equations 1 and 2, and that what I've done is added in the straight angles and subtracted 180 so that now all of the variables a-i are in the expression*)

Now collect together the small triangle angles:

$= (a+b+c)+(d+e+f)+(g+h+i)-360$

(*Look at the equation for what we want, and notice that we need to keep a+b+c. Use inequalities 4 and 5 to get rid of the other angles*)

$(a+b+c)+(d+e+f)+(g+h+i)-360 > (a+b+c)+180+180-360=a+b+c$

Thus

$a+f+g+i>a+b+c$

This is clearly the shortest, but it relies on a certain geometric insight. If you see it, the problem gets easier. Look at the other strategies too though—it's nice to know if you don't have the aha, you can still put together equations and get the right answer.