Oct 9, 2017

An axiom system based on transformations for Euclidean Geometry

Points are objects in a space called the ***plane*** which can be written E^2. A ***line*** is a set of points. Points and lines are otherwise undefined, except that they must satisfy the axioms below:

Line axiom: Given any two points, there is one and only one straight line that contains both points.

Between-ness axiom: If *A, B, C* are three points in a line then exactly one of them is ***between*** the other two.

Distance Axiom: There is a distance function $d: E^2 \times E^2 \to \mathbb{R}$ (where E^2 is the plane), such that

i. $d(P,P)=0$ for every point *P* on the plane.

ii. $d(A,B)>0$ if *A* and *B* are distinct points. — distinct means not the same points

iii. For three distinct points *A, B, C,* $d(A,B)+d(B,C)=d(A,C)$ if and only if *B* lies between *A* and *C* on the line $\overleftrightarrow{AC}$.

iv. For three distinct points *A, B, C,* if *B* does not lie between *A* and *C* or if *B* does not line on $\overleftrightarrow{AC}$, then $d(A,B)+d(B,C)>d(A,C)$

A ***circle*** with center $P \in E^2$ and radius $r \in \mathbb{R}$ is the set of all points $X \in E^2$ such that $d(X,P)=\mathbb{R}$

Separation Axiom: The infinite straight line, the triangle the circle and a pair of distinct rays that share an endpoint, ***separate*** the plane into two regions such that any line or arc joining a point in one region to a point in the other region intersects the separating figure. These regions are called ***sides***. — intersects means shares at least one point

[The definitions of segment and ray are omitted from this list, but are the usual definitions]

An ***angle***, consists of two rays with a common endpoint, and a side of the plane separated by the rays.

Angle Axiom: Given an angle $\angle ABC$, there is a function *m* called the ***measure of the angle*** that maps angles in the plane to real numbered degrees between 0° and 360° inclusive, with the properties that :

i. The trivial angle, consisting of a single ray and itself has measure 0° if the associated region is empty and 360° if the associated region together with the ray comprise the whole plane. Non-trivial angles have measures strictly between 0° and 360°.

ii. Given two angles who share a ray and whose regions do not intersect, the sum of the measures of the angles is the measure of the angle whose sides are the non-shared sides of the angles, and whose region consists of the regions of the two angles and the shared ray.

iii. Given two rays that comprise a line, the measure of the angle is 180°.

Definition: An ***isometry*** is a 1-1, onto function that maps the plane to itself, such that distances and angle measures are preserved. The image of a region under an isometry is called its ***isometric image.*** (If $f(A)=A'$, $f(B)=B'$ and $f(C)=C'$ then distances are preserved if $m(\overline{AB})=m(\overline{A'B'})$, and angle measures are preserved if $m\angle BAC = m\angle B'A'C'$). A point *P* is ***fixed*** by a function *f* if $f(P)=P$.

Isometries Axiom:

i. Given a point (*P*) and an angle ($\angle ABC$), there is a unique isometry (*f*) called a ***rotation*** that fixes the given point and then angle formed by any point, the fixed point, and the image of the point is congruent to the given angle ($\angle XPf(X) = \angle ABC$).

ii. Given a line, there is a unique isometry called a ***reflection*** that fixes points on the line and maps points on one side of the line to the other side of the line.

Definition: Two **segments** are ***congruent*** if they have the same length. Two **angles** are ***congruent*** if they have the same angle measure.

Definition: Two point sets in the plane are ***congruent*** if there is an isometry that maps one to the other. (Named objects are congruent if they are congruent as point sets and if their named parts correspond, eg. $\Delta ABC \cong \Delta DEF$ means that there is an isometry, $f : \Delta ABC \to \Delta DEF$ such that $f(A) = D, f(B) = E, f(C) = F$

Theorem 1 (Sides of Congruent Angles): If two congruent angles share a side (ray) and the other sides (rays) lie on the same side of the shared side, then their other sides are also shared. **I.e.** If $\angle ABC \cong \angle ABD$ and C and D lie on the same side of $\overleftrightarrow{AB}$ then $\overrightarrow{AC} = \overrightarrow{AD}$.

Theorem 2 (Endpoints of Congruent Segments): On a ray, there is at most one point at a given distance from the endpoint of the ray. **I.e.** If $C \in \overrightarrow{AB}$ and $\overline{CA} \cong \overline{BA}$ then $C = B$.

Theorem 3 (Line intersection): Any two distinct lines intersect in at most one point.

Theorem 4: If *f* is an isometry and A, B, C are collinear (on the same line), then $f(A), f(B), f(C)$ are collinear

Theorem 5: If *f* is an isometry, and *X* is on the circle with center *P* and radius *r*, then *f(X)* is on the circle with center *f(P)* and radius *r*.

Theorem 6 (Application): Given triangles ΔABC and ΔDEF there is an isometry that maps A to D, and maps B to a point on $\overrightarrow{DE}$, and maps C to a point to the side of $\overleftrightarrow{DE}$ that contains point F .