Proofs to study

Theorem 1 (Sides of Congruent Angles): If two congruent angles share a side (ray) and the other sides (rays) lie on the same side of the shared side, then their other sides are also shared. **I.e.** If $\angle ABC \cong \angle ABD$ and *C* and *D* lie on the same side of $\overleftrightarrow{AB}$ then $\overrightarrow{BC} = \overrightarrow{BD}$.

Proof	Comments
Let $\angle ABC \cong \angle ABD$ such that (1) C and D lie on the same side of $\overleftrightarrow{AB}$ (2)	State givens (name all objects)
Suppose $\overrightarrow{BC} \neq \overrightarrow{BD}$ (3)	Suppose opposite of conclusion (proof by contradiction set up)
Because C and D lie on the same side of $\overleftrightarrow{AB}$ and $\overrightarrow{BC} \neq \overrightarrow{BD}$, either C lies in the interior of $\angle ABD$ or D lies in the interior of $\angle ABC$ (uses 2,3)	State the possible cases
Without loss in generality, we may assume C lies in the interior of $\angle ABD$ (uses previous line) (4)	You can only do this if the two cases are essentially identical.
Then, by the angle measurement axiom, $m\angle ABC + m\angle CBD = m\angle ABD$ (uses 4) (5)	Key axiom and equation!
But $m\angle ABC = m\angle ABD$ because $\angle ABC \cong \angle ABD$ (uses 1) (6)	
By substitution $m\angle ABC + m\angle CBD = m\angle ABC$ (uses 4 and 5)	
So $m\angle CBD = 0$ (by algebra and previous line) (7)	Givens and algebra get you here
Thus, by the angle measurement axiom $\overrightarrow{BC} = \overrightarrow{BD}$ (uses 7) (8)	Key axiom!
Which contradicts line 3 (8 and 3)	Notice the contradiction
Therefore, $\overrightarrow{BC} = \overrightarrow{BD}$ QED	Here's the conclusion (opposite of "suppose" on line 3)

Theorem 2 (Endpoints of Congruent Segments): On a ray, there is at most one point at a given distance from the endpoint of the ray. **I.e.** If $C \in \overrightarrow{AB}$ and $\overline{CA} \cong \overline{BA}$ then $C = B$.

Proof:

Let $\overrightarrow{AB}$ be a ray (1)

Let $C \in \overrightarrow{AB}$ (2) such that $\overline{CA} \cong \overline{BA}$ (3)

Suppose $C \neq B$ (4)
Thus *A*, *B* and *C* must be distinct points, and by the between-ness axiom, either *C* lies between *A* and *B* or *B* lies between *A* and *C*. (Uses 2 and 4)

Without loss in generality, we may assume that *B* lies between *A* and *C*. (5)

So, by the distance axiom, $d(A,B) + d(B,C) = d(A,C)$ (6)

But $d(A,C) = d(A,B)$ (by 3) (7)

So by algebra $d(A,B) + d(B,C) = d(A,B)$ and $d(B,C) = 0$ (uses 6 and 7) (8)

Thus, by the distance axiom $B = C$ (9)

Which contradicts the assumption that $C \neq B$, so we can conclude that $B = C$

QED

Theorem 3 (Line intersection): Any two distinct lines intersect in at most one point.

Let *l* and *m* be distinct lines.

Suppose *l* and *m* intersect in more than one point.

That means there are at least 2 points in the intersection, so let *A* and *B* be two points in the intersection of *l* and *m*

Now, the line axioms says that there is one and only one line that contains the two points *A* and *B*, and hence *l*=*m*.

This contradicts the given the *l* and *m* are distinct lines.

Therefore, *l* and *m* can intersect in at most one point.

QED

Theorem 4: If *f* is an isometry and A, B, C are collinear (on the same line), then $f(A), f(B), f(C)$ are collinear

proof:

Let *f* be an isometry, and let *A, B*, and *C* be collinear points.

By the between-ness axiom, one of *A, B*, or *C* must lie between the other two.

Without loss in generality, we may assume that *B* lies between *A* and *C*.

Since *B* lies between *A* and *C* on line $\overleftrightarrow{AC}$, by the distance axiom we know that
$d(A,B)+d(B,C)=d(A,C)$ (1)

Because *f* is an isometry, we know that:

$$d(A,B)=d(f(A),f(B))$$

$$d(B,C)=d(f(B),f(C))$$

$$d(A,C)=d(f(A),f(C))$$

Substituting into line 1, we get $d(f(A),f(B))+d(f(B),f(C))=d(f(A),f(C))$

By the distance axiom, since this equation holds, we know that $f(B)$ lies between $f(A)$ and $f(C)$, and all three points are collinear

QED

Theorem 5: If *f* is an isometry, and *X* is on the circle with center *P* and radius *r*, then *f(X)* is on the circle with center *f(P)* and radius *r*.

proof:

Let *f* be an isometry, and let *X* be a point on the circle with center *P* and radius *r*.

Since *X* is on the circle with center *P* and radius *r*, by definition of circle, we know that $d(P,X)=r$ (1)

Since *f* is an isometry we know that $d(P,X)=d(f(P),f(X))$ (2)

By substitution with lines 1 and 2, we know that $d(f(P),f(X))=r$

and hence, by definition of circle, $f(X)$ lies on the circle with center $f(P)$ and radius *r*.