Given a ray with endpoint A, and a point B not on the ray, there is an isometry called a ***rotation*** that fixes A and maps the point B to a point on the ray, with the additional property that the angle between $\overrightarrow{AC}$ and its image has the same measure for every point C in the plane.

**Theorem 10 (Lemma)**: Given two congruent segments $\overline{AB} \cong \overline{DE}$, there is an isometry that maps $A$ to $D$ and $B$ to $E$.

Lemma 10a. Given points $A$ and $D$, there is a point $P$ such that $\overline{PA} \cong \overline{PD}$

Proof: Consider circles $\odot A, \overline{AD}$ and $\odot D, \overline{DA}$
The circles intersect. (why?)
Call one of the points of intersection $P$
Then $\overline{PA} = \overline{PD}$ (why?)

Lemma 10b. There is an isometry that maps $A$ to $D$,

Proof: By Lemma 10a, there is a point $P$ such that $\overline{PA} \cong \overline{PD}$
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation with fixed point $P$ that maps $A$ to a point on the ray $\overrightarrow{PD}$
Let $f(A) = A'$
Then $A' \in \overrightarrow{PD}$

Then $m(PA) = m(f(P)f(A)) = m(PA')$ (why?)

So $m(PD) = m(PA')$ (why?)

So by theorem 2, $A' = D$ (check hypotheses)
And so the isometry $f$ maps $A$ to $D$.

Proof: By lemma 10b, there is an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(A) = D$
Let $f(B) = B'$
Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation with fixed point $D$ that maps $B'$ to a point on the ray $\overrightarrow{DE}$
Let $f(B') = B''$
Then $B'' \in \overrightarrow{DE}$
And $m(DB'') = m(DE)$ (why?)

So by theorem 2, $B'' = E$ (check hypotheses)

By theorem 9, $g \circ f$ is an isometry.
$g \circ f(A) = D$ (why?)
$g \circ f(B) = E$ (why?)
Thus $g \circ f$ is an isometry that maps A to D and B to E

**Theorem 11 (Lemma)**: Given triangles $\Delta ABC$ and $\Delta DEF$ such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle ABC \cong \angle DEF$, there is an isometry that maps *A* to *D*, *B* to *E* and *C* to *F*.

Lemma 11a: Given triangles $\Delta ABC$ such that $\overline{AB} \cong \overline{DE}$, there is an isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*.
by Lemma 10, there is an isometry $f:\mathbb{R}^2 \to \mathbb{R}^2$ such that $f(A) = D$ and $f(B) = E$
Let $f(C) = C'$
Since *C* is not on line $\overleftrightarrow{AB}$, $C'$ is not on line $\overleftrightarrow{DE}$ by theorem 5.

| case 1: $C'$ is on the same side of $\overleftrightarrow{DE}$ as *F*. | case 2: $C'$ is on the opposite side of $\overleftrightarrow{DE}$ from *F*. |
|---|---|
| Then *f* is an isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*. | Let $g:\mathbb{R}^2 \to \mathbb{R}^2$ be the reflection that fixes $\overleftrightarrow{DE}$<br>Then $g(D) = D$<br>and $g(E) = E$<br>Let $g(C') = C''$<br>Then $C''$ is on the same side of $\overleftrightarrow{DE}$ as *F*.<br>So $g \circ f$ is an isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*. |

proof of theorem 11:

By lemma 11a, there is an isometry $f:\mathbb{R}^2 \to \mathbb{R}^2$ isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*.
Let $f(C) = C'$
Then $C'$ is on the same side of $\overleftrightarrow{DE}$ as *F*.
$m(\angle C'ED) = m(\angle FED)$ (why?)

By theorem 1, $C' \in \overrightarrow{ED}$ (check hypotheses)

$m(EC') = m(EF)$ (why?)

By theorem 2, $C' = F$

Hence, *f* is an isometry that maps *A* to *D*, *B* to *E* and *C* to *F*.