Hyperbolic Geometry: Bolyai, Lobachevsky and Gauss

Hyperbolic and Euclidean lines facts:

From these agreed upon axioms or assumptions:

I. Given any two points there is exactly one straight line that contains them

II. Angles and distances can be measured and duplicated

III. Triangles that have congruent 2 sides and the angle between them (SAS) are congruent (the other side and angles are also congruent)

Euclid proved that

IV. If you have a pair of lines and a transversal with alternate interior angles being equal, then the lines have to be parallel.

This is without the parallel postulate, and it is true in the Euclidean plane and in the Hyperbolic plane

In Euclidean geometry, with the Euclidean parallel postulate, it's possible to prove that *every* pair of parallel lines and transversal has this same property (congruent alternate interior angles), but in Hyperbolic geometry, there are parallel lines and transversals where the alternate interior angles are different.

Euclidean geometry: Add in the assumption/axiom:

V. The sum of angles in any triangle is 180°.

1. In this triangle ΔABC , consider the line $\overleftrightarrow{BC}$ that includes side $\overline{BC}$.

- We can use II to make a line $\overleftrightarrow{AE}$ such that the alternate interior angles shown in figure 1 are equal. Theorem IV tells us that $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AE}$.
- We can also use II to make a line $\overleftrightarrow{AD}$ such that the alternate interior angles shown in figure 2 are equal. Theorem IV tells us that $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AD}$.

Use V to prove that the angles $\angle EAB$, $\angle BAC$ and $\angle DAC$ add up to $180°$. If you can prove that, it means that *E*, *A* and *D* are all on the same line and $\overleftrightarrow{AE} = \overleftrightarrow{AD}$

Hyperbolic Geometry: Instead of V, add in the assumption:

V′. The sum of angles in any triangle is strictly less than 180°.

2. In this triangle ΔABC , consider the line $\overleftrightarrow{BC}$ that includes side $\overline{BC}$.

- We can use II to make a line $\overleftrightarrow{AE}$ such that the alternate interior angles shown in figure 3 are equal. Theorem IV tells us that $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AE}$.
- We can also use II to make a line $\overleftrightarrow{AD}$ such that the alternate interior angles shown in figure 4 are equal. Theorem IV tells us that $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{AD}$.

Use V′ to prove that the angles $\angle EAB$, $\angle BAC$ and $\angle DAC$ do not add up to $180°$. If you can prove that, it means that *E*, *A* and *D* are not all on the same line and $\overleftrightarrow{AE} \neq \overleftrightarrow{AD}$.

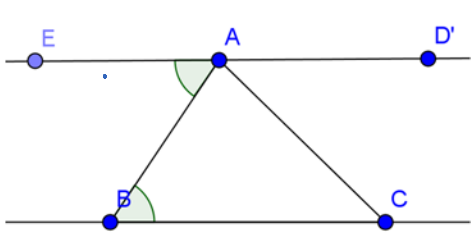

figure 1

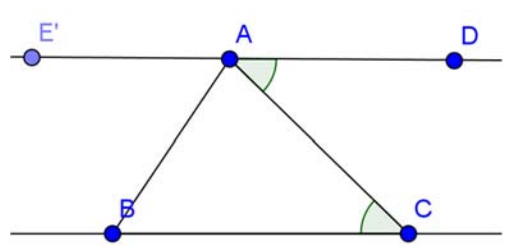

figure 2