Theorem 9: The composition of two isometries is an isometry.

Given: $f:\mathbb{R}^2 \to \mathbb{R}^2$ and $g:\mathbb{R}^2 \to \mathbb{R}^2$ are isometries

Then: $f \circ g$ is an isometry

Lemma 10: Given points *A* and *D*, there is a point *P* such that $\overline{PA} \cong \overline{PD}$

Lemma 11: Given points *A* and *D*, there is an isometry that maps *A* to *D.*

Lemma 12: Given two congruent segments $\overline{AB} \cong \overline{DE}$, there is an isometry that maps *A* to *D* and *B* to *E*.

Lemma 13: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, there is an isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*.

Lemma 14: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle ABC \cong \angle DEF$, there is an isometry that maps *A* to *D*, *B* to *E* and *C* to *F*.

Lemma 15: If two triangles are isometric, then they are congruent.

Given: ΔABC and ΔDEF and an isometry *f* such that $f(A) = A', f(B) = B', f(C) = C'$

Then: $\Delta ABC \cong \Delta DEF$

Theorem 16 (SAS): If two sides of one triangle are congruent to two corresponding sides of another triangle, and the angles contained by those sides are congruent, then the triangles are congruent.

Given: ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle ABC \cong \angle DEF$

Then: $\Delta ABC \cong \Delta DEF$

Lemma 17: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, and an isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*, then the image of *C* lies on $\overrightarrow{EF}$.

Lemma 17 alternate: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, then there is an isometry that maps *A* to *D* and *B* to *E* such that *C* is mapped to a point on the same side of $\overleftrightarrow{DE}$ as *F*, and the image of *C* lies on $\overrightarrow{EF}$.

Lemma 18: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$ and $\angle BAC \cong \angle EDF$ then there is an isometry that maps *A* to *D*, *B* to *E* and *C* to *F*

Theorem 19 (ASA): If two angles of one triangle and the segment between them are congruent to the corresponding angles and side of another triangle then the triangles are congruent.

Given:

Then:

Theorem 20: (segments can be duplicated): Given a ray starting at a given point, and a distance between two points, there exists another point that lies on the given ray such that the segment between it and the starting point of the ray is equal to the given distance.

Given:

Then:

Lemma 21: Given $\angle ABC$ and ray $\overrightarrow{DE}$ there is an isometry that maps *B* to *D*, and *A* to a point on ray $\overrightarrow{DE}$

Lemma 22: Given a non-trivial angle $\angle ABC$, ray $\overrightarrow{DE}$ and a side of $\overleftrightarrow{DE}$, then there is an isometry that maps *B* to *D*, *A* to a point on ray $\overrightarrow{DE}$, and *C* to a point on the given side of $\overleftrightarrow{DE}$

Theorem 23 (angles can be duplicated): Given a non-trivial angle, a ray, and a side of the line containing the ray, there exists another ray such that the rays together are congruent to the given angle.

Given:

Then:

Definition: Two point sets in the plane are ***congruent*** if there is an isometry that maps one to the other.

A triangle is **Isosceles** if (at least) two sides are congruent. In an isosceles triangle, the **base** refers to the side which is not identified as one of the two congruent sides, the base angles are the angles which share a side with the base, and the vertex angle is the angle between the two congruent sides.

Lemma 24: Given triangle ΔABC such that $\overline{AB} \cong \overline{AC}$ then there is an isometry that fixes *A* and maps *B* to *C* and *C* to *B*.

Theorem 24 (equal sides implies equal angles): In an isosceles triangle, the base angles are congruent.

Given:

Then:

Theorem 25 (equal angles implies equal sides): In a triangle, if two angles are congruent, then the sides opposite those angles are also congruent.

Given:

Then:

Lemma 26: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, then there exists an isometry that maps *A* to *D* and *B* to *E* and maps *C* to the opposite side of $\overleftrightarrow{DE}$ from *F*.

Lemma 27: Given triangles ΔDEF and ΔDEG such that $\overline{FD} \cong \overline{GD}$ and $\overline{FE} \cong \overline{GE}$ and $\overline{FG}$ intersects $\overline{DE}$ at a point that is neither *D* nor *E*, then $\angle DFE \cong \angle DGE$

Lemma 28: Given triangles ΔDEF and ΔDEG such that $\overline{FD} \cong \overline{GD}$ and $\overline{FE} \cong \overline{GE}$ and $D \in \overline{FG}$, then $\angle DFE \cong \angle DGE$