Oct 25 2016

An axiom system based on transformations for Euclidean Geometry

Axiom 1 (lines): Given any two points, there is one and only one straight line that contains both points.
Note: lines are sets of points

Axiom 2 (Separation and between-ness): The infinite straight line, the triangle and the circle ***separate*** the plane into two regions such that any line or arc of a circle joining a point in one region to a point in the other region intersects the separating figure. These regions are called ***sides***.

Definitions: A point on an infinite straight line divides the line into two regions, where the point lies ***between*** the regions and between pairs of points (one in each region). Given any three points on a line, exactly one of the points will lie between the other two. A point on a line, together with one of the sides of the point on the line is a ***ray***. A ***segment*** between two points called endpoints consists of the two endpoints and all of the points on the line containing the endpoints that lie between the two points.

Axiom 3 (Existence of lengths and angles):

i. **Triangle Inequality** Given any two points A and B, there is a non-negative real number that is called the ***distance*** between them, with the properties that:
  - *a.* Any point is distance 0 from itself, and is a positive distance from any other point.
  - *b.* If C is a point on a segment, then the sum of the distances from the endpoints to C is equal to the distance between the endpoints.
  - *c.* If C is a point not on a given segment, then the sum of the distances from the endpoints to C is greater than the distance between the endpoints.

ii. Given an ***angle***, consisting of two rays with a common endpoint, and a choice of the regions separated by the two rays, then there is a real number between 0° and 360° that is called the ***measure of the angle***, with the properties that :
  - *a.* The trivial angle, consisting of a single ray and itself has measure 0° if the associated region is empty and 360° if the associated region together with the ray comprise the whole plane. Non-trivial angles have measures strictly between 0° and 360°.
  - *b.* Given two angles who share a ray and whose regions do not intersect, the sum of the measures of the angles is the measure of the angle whose sides are the non-shared sides of the angles, and whose region consists of the regions of the two angles and the shared ray.
  - *c.* Given two rays that comprise a line, the measure of the angle is 180°.

*Definition***:** An ***isometry*** (also called a ***rigid motion***) is a 1-1, onto function that maps the plane to itself in such a way that distances and angle measures are preserved. The image of a region under an isometry is called an ***isometric image*.** (I.e. if f is an isometry and $f(A)=A'$, $f(B)=B'$ and $f(C)=C'$ then $m(\overline{AB})=m(\overline{A'B'})$, $m\angle BAC = m\angle B'A'C'$, if $A'=B'$ then $A=B$, and if P is any point, then there is another point Q such that $f(Q)=P$). A point P is ***fixed*** by a function f if $f(P)=P$.

Axiom 4 (Existence of isometries):

i. Given a line, there is one isometry called a ***reflection*** that fixes points on the line and maps one side of the line to the other side of the line.

ii. Given a ray with endpoint A, and a point B not on the ray, there is an isometry called a ***rotation*** that fixes A and maps the point B to a point on the ray, with the additional property that the angle between $\overrightarrow{AC}$ and its image has the same measure for every point C in the plane.

Definition: Two geometric objects are ***distinct*** if they are not equal as point sets (they do not have all the same points as each other). For the purposes of this definition we will treat a point as a set consisting of a single point. Two geometric objects ***intersect*** if they have at least one point in common.

Definition: Two **segments** are **congruent** if they have the same *length*. Two **angles** are **congruent** if they have the same *angle measure*. Two **triangles** are congruent if they have respectively the **same side lengths and angle measures**.

Theorem 1 (Sides of Congruent Angles): If two congruent angles share a side (ray) and the other sides (rays) lie on the same side of the shared side, then their other sides are also shared. **I.e.** If $\angle ABC \cong \angle ABD$ and C and D lie on the same side of $\overleftrightarrow{AB}$ then $\overrightarrow{AC} = \overrightarrow{AD}$.

Theorem 2 (Endpoints of Congruent Segments): On a ray, there is at most one point at a given distance from the endpoint of the ray. **I.e.** If $C \in \overrightarrow{AB}$ and $\overline{CA} \cong \overline{BA}$ then $C = B$.

Theorem 3 (Line intersection): Any two distinct lines intersect in at most one point.

Theorem 4: The isometric image of a line segment is a line segment.

Given:

Then:

Theorem 5: The isometric image of a line is a line.

Given:

Then

Definition: A ***circle*** with a given center and radius length is the set of all points whose distance from the center is equal to the given length.

Theorem 6: The isometric image of a circle is a circle with the same radius.

Given:

Then:

Defn: A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one if the pre-image of any point is at most one point.

Given:

Then:

Theorem 7: The composition of two 1-1 functions is 1-1.

Defn: A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is onto if every point in $\mathbb{R}^2$ is the image of a point.

Given:

Then:

Theorem 8: The composition of two onto functions is onto.

Given:

Then:

Theorem 9: The composition of two isometries is an isometry.

Given:

Then:

Lemma 10: Given points A and D, there is a point P such that $\overline{PA} \cong \overline{PD}$

Lemma 11: Given points A and D, there is an isometry that maps A to D.

Lemma 12: Given two congruent segments $\overline{AB} \cong \overline{DE}$, there is an isometry that maps A to D and B to E.

Lemma 13: Given triangles ΔABC such that $\overline{AB} \cong \overline{DE}$, there is an isometry that maps A to D and B to E such that C is mapped to a point on the same side of $\overleftrightarrow{DE}$ as F.

Lemma 14: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle ABC \cong \angle DEF$, there is an isometry that maps A to D, B to E and C to F.

Lemma 15: If two triangles are isometric, then they are congruent.

Given:

Then:

Theorem 16 (SAS): If two sides of one triangle are congruent to two corresponding sides of another triangle, and the angles contained by those sides are congruent, then the triangles are congruent.

Given:

Then:

Lemma 17: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, and an isometry that maps A to D and B to E such that C is mapped to a point on the same side of $\overleftrightarrow{DE}$ as F, then the image of C lies on $\overrightarrow{EF}$.

Lemma 18: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$ then there is an isometry that maps A to D, B to E and C to F

Theorem 19 (ASA): If two angles of one triangle and the segment between them are congruent to the corresponding angles and side of another triangle then the triangles are congruent.

Given:

Then:

Theorem 15: (segments can be duplicated): Given a ray starting at a given point, and a distance between two points, there exists another point that lies on the given ray such that the segment between it and the starting point of the ray is equal to the given distance.

Given:

Then: