Theorem 1 (Line intersection): Any two distinct lines intersect in at most one point.

Given: lines l and m, where *l* and *m* are not the same line.
To prove: The intersection of *l* and *m* contains no more than one point.

Proof: Suppose the intersection of *l* and *m* contains more than one point.
Then it contains at least 2 points.
Let *A* and *B* be two points in the intersection of *l* and *m* .
By axiom 1, there is one and only one line that contains both *A* and *B*, so *l*=*m* is the line that contains both *A* and *B*.
This contradicts the assumption that *l* and *m* are distinct.
Hence the intersection contains no more than one point.
QED

Theorem 2 (Endpoints of Congruent Segments): If two congruent segments lie on the same ray, and share an endpoint with the ray, then their other endpoints are the same (and the segments are equal as point sets).

Given: $\overline{AB} \cong \overline{AC}$ such that $\overrightarrow{AC} = \overrightarrow{AB}$
To prove: *C*=*B*.
Proof: Suppose $C \neq B$
Then either *C* is between *A* and *B* or *B* is between *A* and *C*.
Note: one could now prove both cases, or combine both cases into one case:

Case 1: suppose *C* is between *A* and *B* Then $m(\overline{AC}) + m(\overline{CB}) = m(\overline{AB})$ (measurement axiom) But $m(\overline{AC}) = m(\overline{AB})$ (given) So $m(\overline{CB}) = 0$ By the measurement axiom *C*=*B*.	Case 2: suppose *B* is between *A* and *C* Then $m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC})$ (measurement axiom) But $m(\overline{AB}) = m(\overline{AC})$ (given) So $m(\overline{BC}) = 0$ By the measurement axiom *C*=*B*

QED.

Theorem 3 (Sides of Congruent Angles): If two congruent angles share a side and whose other sides lie on the same side of the shared side, then their other sides are also shared, and the angles are identically equal as point sets.

Given $\angle ABC \cong \angle ABD$ such that *C* and *D* lie on the same side of $\overleftrightarrow{AB}$
To prove: $\overrightarrow{BC} = \overrightarrow{BD}$

Proof:
Suppose $\overrightarrow{BC} \neq \overrightarrow{BD}$
Then either C lies in the angle $\angle ABD$ or *D* lies in the angle $\angle ABC$
Note: one could now prove both cases, or combine both cases into one case:
Without loss in generality, we may assume that C lies in the angle $\angle ABD$
Then $m\angle ABC + m\angle CBD = m\angle ABD$ (measurement axiom)
But $m\angle ABC = m\angle ABD$ (given)
So $m\angle CBD = 0$
Then by the measurement axiom $\overrightarrow{BC} = \overrightarrow{BD}$
QED

***Theorem 4 (Triangle Inequality):** If the sum of the distances from the endpoints of a segment to a point is equal to the distance between the endpoints of the segment, then the point lies on the segment.

Given a segment $\overline{AB}$ and a point *C*, such that $m(\overline{AC})+m(\overline{CB})=m(\overline{AB})$

To prove: $C\in\overline{AB}$

Proof: Suppose $C\notin\overline{AB}$

Then by the measurement axiom, $m(\overline{AC})+m(\overline{CB})>m(\overline{AB})$

Which contradicts the given that $m(\overline{AC})+m(\overline{CB})=m(\overline{AB})$

Hence $C\in\overline{AB}$

QED

Theorem 5: The isometric image of a line segment is a line segment with the same length.

Given: a line segment $\overline{AB}$ and an isometry $f:E^2\to E^2$

To prove: $f(\overline{AB})=\overline{f(A)f(B)}$ and $m(\overline{AB})=m(f(\overline{AB}))$

Proof: Let $C\in\overline{AB}$

By the measurement axiom, $m(\overline{AC})+m(\overline{CB})=m(\overline{AB})$

Since *f* is an isometry, $m(\overline{AC})=m(\overline{f(A)f(C)})$, $m(\overline{CB})=m(\overline{f(C)f(B)})$ and $m(\overline{AB})=m(\overline{f(A)f(B)})$

So, by substitution $m(\overline{f(A)f(C)})+m(\overline{f(C)f(B)})=m(\overline{f(A)f(B)})$

By the triangle inequality, $f(C)\in\overline{f(A)f(B)}$

This means that any point in the segment $\overline{AB}$ *maps to a point in the segment* $\overline{f(A)f(B)}$

Thus $f(\overline{AB})\subset\overline{f(A)f(B)}$ (uh-oh—maybe this isn't done yet)

Other half: Let *D* be a point on $\overline{f(A)f(B)}$.

Since *f* is an isometry, there is a point *E* such that $f(E)=D$

By the triangle inequality $m(\overline{f(A)D})+m(\overline{Df(B)})=m(\overline{f(A)f(B)})$

Since *f* is an isometry, $m(\overline{AE})=m(\overline{f(A)D})$, $m(\overline{EB})=m(\overline{Df(B)})$ and $m(\overline{AB})=m(\overline{f(A)f(B)})$

So, by substitution, $m(\overline{AE})+m(\overline{EB})=m(\overline{AB})$

This means every point on $\overline{f(A)f(B)}$ *is the image of a point on* $\overline{AB}$

Hence $f(\overline{AB})\supset\overline{f(A)f(B)}$

Thus $f(\overline{AB})=\overline{f(A)f(B)}$

Since *f* is an isometry, $m(\overline{AB})=m(\overline{f(A)f(B)})=m(f(\overline{AB}))$ QED

Theorem 6: The isometric image of a line is a line.

Given a line $\overleftrightarrow{AB}$ and an isometry $f:E^2\to E^2$

To prove: $f(\overleftrightarrow{AB})=\overleftrightarrow{f(A)f(B)}$

Proof: Let $C\in\overleftrightarrow{AB}$

Then one of *A*, *B* or *C* lies between the other two

Case 1: $C\in\overline{AB}$

Then by Theorem 5, $f(C) \in \overline{f(A)f(B)} \subset \overleftrightarrow{f(A)f(B)}$

Case 2: $B \in \overline{AC}$

Then by Theorem 5, $f(B) \in \overline{f(A)f(C)} \subset \overleftrightarrow{f(A)f(C)}$

Then $\overleftrightarrow{f(A)f(C)} = \overleftrightarrow{f(A)f(B)}$

And thus $f(C) \in \overleftrightarrow{f(A)f(B)}$

Case 3: Similarly if $A \in \overline{BC}$, $f(C) \in \overleftrightarrow{f(A)f(B)}$

Thus $f(\overleftrightarrow{AB}) \subset \overleftrightarrow{f(A)f(B)}$

Let $D \in \overleftrightarrow{f(A)f(B)}$

Since *f* is an isometry, there is a point *E* such that $f(E) = D$

then one of *D*, *f(A)*, or *f(B)* lies between the other two

Case 1: $D \in \overline{f(A)f(B)}$

Then by theorem 5, $E \in \overline{AB}$

Case 2: $f(A) \in \overline{Df(B)}$

By theorem 5, $A \in \overline{EB} \subset \overleftrightarrow{EB}$

Thus $\overleftrightarrow{EB} = \overleftrightarrow{AB}$ and so $E \in \overline{AB}$

Case 3: Similarly, if $f(B) \in \overline{Df(A)}$ then $E \in \overline{AB}$

So $f(\overleftrightarrow{AB}) \supset \overleftrightarrow{f(A)f(B)}$

Hence $f(\overleftrightarrow{AB}) = \overleftrightarrow{f(A)f(B)}$

Definition: A ***circle*** with a given center and radius length is the set of all points whose distance from the center is equal to the given length.

Theorem 7: The isometric image of a circle is a circle with the same radius.

Given a circle *C* with center *A* and radius *r* and an isometry *f*

To prove: $f(C)$ is a circle with center $f(A)$ and radius *r*.

Proof: Let C' be the circle with center $f(A)$ and radius *r*.

Let $P \in C$

Then $m(\overline{PA}) = r$

Since *f* is an isometry, $m(\overline{f(P)f(A)}) = m(\overline{PA}) = r$

So $f(P) \in C'$

Thus $f(C) \subset C'$

Let *Q* be a point on the circle with center $f(A)$ and radius *r*.

Then $m(\overline{Qf(A)}) = r$

Because *f* is an isometry, there is a point *R* such that $f(R) = Q$

So, $m(\overline{RA}) = m(\overline{Qf(A)}) = r$ and hence *R* is in *C*.

Thus $f(C) \supset C'$

And hence $f(C) = C'$ QED