Parallel line theorems:

There are two parallel line theorems: one which can use used to prove that lines are parallel, and another which can be used to deduce angle properties of lines that you already know are parallel.

The first theorem: the one that lets you construct parallel lines, needs only the fact that two lines can intersect in only one point and some congruent triangle constructions and theorems. This theorem (that tells you lines are parallel if they have some angle properties) is true in both Euclidean and Hyperbolic geometry.

The second theorem: the one that tells you that parallel lines always have certain angle properties, is true only in Euclidean geometry. In Hyperbolic geometry, parallel lines can have lots of different angle properties. You need an axiom to prove this one (our Axiom 5) that says that if the angle properties aren't true then the lines aren't parallel.

Tip: Don't confuse the property of having a transversal line with the property of being parallel. We're using transversal to *only* mean that the line crosses both of the other lines it doesn't say anything about whether the other lines cross each other or not. By saying that a line is a transversal, we are saying that when we talk about alternate interior angles and corresponding angles we mean the appropriate angles to the line we are calling the transversal.

Theorem 19 (parallels exist): Given two lines and a transversal line that intersects both lines, if the alternate interior angles are congruent, then the lines are parallel.

Proof: Let l and m be lines and let t be a transversal line which intersects both l and m at points A and B respectively such that a pair of alternate interior angles $\angle b$ and $\angle c$ are congruent, where $\angle b$ has vertex A and $\angle c$ has vertex B. **

We will call the other pair of alternate interior angles $\angle a$ and $\angle d$, where $\angle a$ and $\angle d$ are on a line with to $\angle b$ and $\angle c$ respectively. **

Suppose that lines l and m intersect. Call the point of intersection C, then there is a triangle with vertices A, B and C. One of the angles $\angle b$ and $\angle c$ is an angle interior to ΔABC; without loss in generality, we may assume that it is $\angle b$.

There exists a point D on line m such that D lies on the opposite side of t from C and $\overline{BD} \cong \overline{AC}$ (Thm. 10)

Note that:

- $\angle CAB = \angle b \cong \angle c = \angle ABD$ (given)
- $\overline{AC} \cong \overline{BD}$ (construction)
- $\overline{AB} \cong \overline{AB}$ (same segment

Hence, $\Delta ABC \cong \Delta BAD$ (SAS)
Thus, $\angle DAB \cong \angle ABC = \angle d$ (CPCTC)

Now $m\angle c = m\angle b$ (given) and $m\angle b + m\angle a = 180°$ and $m\angle c + m\angle d = 180°$ (angles on a straight line)
By substitution and algebraic equivalence

$m\angle b + m\angle a = 180°$	$m\angle c + m\angle d = 180°$
$m\angle c + m\angle a = 180°$	$m\angle d = 180° - m\angle c$
$m\angle a = 180° - m\angle c$	

And hence $m\angle a = m\angle d$ (by transitivity or substitution)

Since $\angle DAB \cong \angle d$ and $m\angle a = m\angle d$, we get $\angle DAB \cong \angle a$ (transitivity/substitution)

Now, $\angle DAB$ and $\angle a$ share a side $\overrightarrow{AB}$ and the angles lie on the same side of t (they are both opposite C), so by theorem 3, $\overrightarrow{AD}$ is the other side of $\angle a$, which means that $D \in l$

D was constructed to be on m, so $D \in l \cap m$.
But $C \in l \cap m$
This contradicts axiom 1, which says that two line intersect in at most one point.
Thus we conclude that our supposition (that l and m intersect) is false, and hence we conclude that l and m have no point of intersection, and they are parallel.

Theorem 22: Given parallel straight lines and a transversal intersecting the two lines, then the interior angles on the same side of the transversal have a sum of 180°.

Proof: Let l and m be parallel straight lines, and let t be a transversal that intersects l and m at points A and B respectively. Let angles $\angle a$, $\angle b$, $\angle c$ and $\angle d$ be the interior angles along t with the properties that:

- $\angle a$ and $\angle b$ form a straight line at A,
- $\angle c$ and $\angle d$ form a straight line at B,
- the pair $\angle a$ and $\angle c$ lies on the same side of t as each other,
- the pair $\angle b$ and $\angle d$ lie on the same side of t as each other. **

The sum of $m\angle a + m\angle c$ is either equal to 180°, less than 180° or greater than 180°.

Case 1: Suppose $m\angle a + m\angle c < 180°$

By Axiom 5, then lines l and m intersect (on the same side of t as angles $\angle a$ and $\angle c$), which contradicts the given property that l and m are parallel. Hence we may conclude that $m\angle a + m\angle c \nless 180°$.

Case 2: Suppose $m\angle a + m\angle c > 180°$

We can say $m\angle a + m\angle b = 180°$ and $m\angle c + m\angle d = 180°$ (angles on a straight line)
By algebraically manipulating these equations, we get:
$m\angle a = 180° - m\angle b$ and $m\angle c = 180° - m\angle d$
Substituting into the inequality, we get:
$180° - \angle b + 180° - \angle d > 180°$
Which is equivalent to:
$180° > \angle b + \angle d$
By Axiom 5, then lines l and m intersect (on the same side of t as angles $\angle b$ and $\angle d$), which contradicts the given property that l and m are parallel. Hence we may conclude that $m\angle a + m\angle c \ngtr 180°$.

Since $m\angle a + m\angle c \nless 180°$ and $m\angle a + m\angle c \ngtr 180°$ it must be the case that $m\angle a + m\angle c = 180°$

**Note: if you don't want to write down a list of which are corresponding angles and which make a straight line when defining your notation, you can provide a diagram and say something like: "angles with the relative positions shown in the diagram". *You must do something to define your angles before you use them in your proof*.