Euclidean Plane Geometry versions:

# Hilbert's version:

## I. Incidence

1. For every two points $A$ and $B$ there exists a line $a$ that contains them both
2. For every two points there exists no more than one line that contains them both.
3. There exist at least two points on a line. There exist at least three points that do not lie on a line.

## II. Order

1. If a point $B$ lies between points $A$ and $C$, $B$ is also between $C$ and $A$, and there exists a line containing the distinct points $A,B,C$.
2. If $A$ and $C$ are two points of a line, then there exists at least one point $B$ lying between $A$ and $C$.
3. Of any three points situated on a line, there is no more than one which lies between the other two.
4. Pasch's Axiom: Let $A$, $B$, $C$ be three points not lying in the same line and let $a$ be a line lying in the plane $ABC$and not passing through any of the points $A$, $B$, $C$. Then, if the line $a$ passes through a point of the segment$AB$, it will also pass through either a point of the segment $BC$ or a point of the segment $AC$.

## III. Congruence

1. If $A$, $B$ are two points on a line $a$, and if $A'$ is a point upon the same or another line $a'$ , then, upon a given side of $A'$ on the straight line $a'$ , we can always find a point $B'$ so that the segment $AB \cong A'B'$ . Every segment is congruent to itself; $AB \cong AB$.
2. If $AB \cong A'B'$ and $AB \cong A''B''$, then $A'B' \cong A''B''$.
3. Let $AB$ and $BC$ be two segments of a line $a$ which have no points in common aside from the point $B$, and, let $A'B'$ and $B'C'$ be two segments of the same or of another line $a'$ having, no point other than $B'$ in common. Then, if $AB \cong A'B'$ and $BC \cong B'C'$, we have $AC \cong A'C'$.
4. Let an angle $\angle (h,k)$ be given in the plane α and let a line $a'$ be given in a plane α′. Suppose also that, in the plane α′, a definite side of the straight line $a'$ be assigned. Denote by $h'$ a ray of the straight line $a'$ emanating from a point $O'$ of this line. Then in the plane α′ there is one and only one ray $k'$ such that the angle $\angle (h, k)$, or $\angle (k, h)$, is congruent to the angle $\angle (h', k')$ and at the same time all interior points of the angle $\angle (h', k')$ lie upon the given side of $a'$.
5. If $\angle (h, k) \cong \angle (h', k')$ and $\angle (h, k) \cong \angle (h'', k'')$, then $\angle (h', k') \cong \angle (h'', k'')$.
6. If, in the two triangles ABC and A′B′C′ the congruences $AB \cong A'B'$, $AC \cong A'C'$, $\angle BAC \cong \angle B'A'C'$ hold, then the congruence $\angle ABC \cong \angle A'B'C'$ holds.

## IV. Parallels

1. (Playfair's Axiom) Let $a$ be any line and $A$ a point not on it. Then there is at most one line in the plane, determined by $a$ and $A$, that passes through $A$ and does not intersect $a$.

## V. Continuity

1. Axiom of Archimedes. If $AB$ and $CD$ are any segments then there exists a number $n$ such that $n$ segments $CD$ constructed contiguously from $A$, along the ray from $A$ through $B$, will pass beyond the point $B$.
2. *Axiom of line completeness*. An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III and from V-1 is impossible.

# Euclid's Axioms

### Postulates

Let the following be postulated:

**Postulate 1.**
To draw a straight line from any point to any point.

**Postulate 2.**
To produce a finite straight line continuously in a straight line.

**Postulate 3.**
To describe a circle with any center and radius.

**Postulate 4.**
That all right angles equal one another.

**Postulate 5.**
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

### Common Notions

**Common notion 1.**
Things which equal the same thing also equal one another.

**Common notion 2.**

If equals are added to equals, then the wholes are equal.

**Common notion 3.**
If equals are subtracted from equals, then the remainders are equal.

**Common notion 4.**
Things which coincide with one another equal one another.

**Common notion 5.**
The whole is greater than the part.

# Birkhoff's Axioms:

**Postulate I: Postulate of Line Measure**. A set of points $\{A, B, ...\}$ on any line can be put into a 1:1 correspondence with the real numbers $\{a, b, ...\}$ so that $|b - a| = d(A, B)$ for all points $A$ and $B$.

**Postulate II: Point-Line Postulate**. There is one and only one line, $\ell$, that contains any two given distinct points $P$and $Q$.

**Postulate III: Postulate of Angle Measure**. A set of rays $\{\ell, m, n, ...\}$ through any point $O$ can be put into 1:1 correspondence with the real numbers $a$ (mod $2\pi$) so that if $A$ and $B$ are points (not equal to $O$) of $\ell$ and $m$, respectively, the difference $a_m - a_\ell$ (mod $2\pi$) of the numbers associated with the lines $\ell$ and $m$ is $\angle AOB$. Furthermore, if the point $B$ on $m$ varies continuously in a line $r$ not containing the vertex $O$, the number $a_m$ varies continuously also.

**Postulate IV: Postulate of Similarity**. Given two triangles $ABC$ and $A'B'C'$ and some constant $k > 0$, $d(A', B') = kd(A, B)$, $d(A', C') = kd(A, C)$ and $\angle B'A'C' = \pm\angle BAC$, then $d(B', C') = kd(B, C)$, $\angle C'B'A' = \pm\angle CBA$, and $\angle A'C'B' = \pm\angle ACB$.