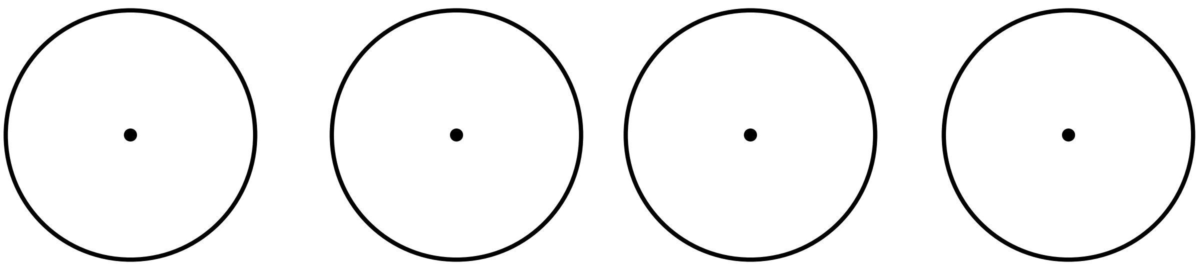
Fractions, Ratios and the Number system:

Models: We're going to use three of the most common models or visualizations for fractions: geometric fractions of a circle and a rectangle, and number line fractions of a length. Each of these models has advantages and disadvantages for learning about fractions. In grade 2, students are introduced to fractions using geometric shapes (such as circles and rectangles) and in grade 3, students are introduced to fractions on a number line.

Most of the visualizations we draw for fractions, are built on an area model of a fraction: the relationship between the area of the part and the area of the whole represents the fraction.

In a **circle** model, the circle represents the whole. The circle is subdivided using radii (segments from the center of the circle to the perimeter), that cut the circle into sectors with equal area. These sectors represent the fractional amount when compared to a whole that is represented by the entire circle.

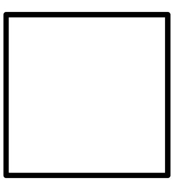
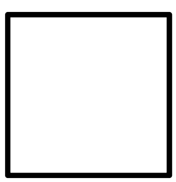
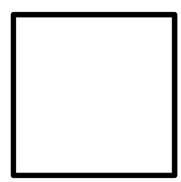
A circle model of fractions has the advantage that our brains are pretty good at recognizing and estimating angle sizes, so the visualization of the size of a third or a fourth or a fifth is something we're pretty good at remembering and estimating. Practice your estimation strategies on these circles by subdividing them:



thirds fourths fifths sixths

In a **rectangular** model, a rectangle or square (a square is just a special kind of rectangle) represents the whole. The whole is then subdivided using horizontal and vertical lines to make equal sized parts. Rectangles and squares can also be divided into halves or fourths using diagonal lines, but those aren't going to give us the most useful properties that rectangular models have, so we're going to just use subdividing lines that are parallel to the sides.

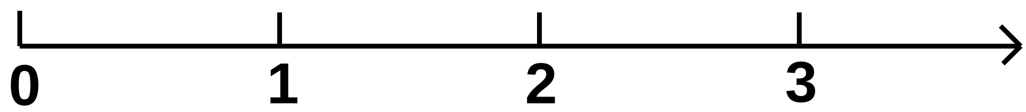
Rectangular models get really useful when you are making subdivisions in both directions at once. For instance, you could subdivide these squares square into sixths in two ways: either by making all of the subdivisions parallel to the same side, or by making some dividing lines vertical and others horizontal.

 Show sixths using only vertical lines	 Show sixths using some vertical and some horizontal lines	In the pictures of sixths you just drew, can you see thirds in the same picture? Can you see halves?	 How would you draw 20ths on a rectangular diagram?

The **number line** model of fractions shows a fraction as a length. The length of a whole unit is marked and labelled as 1 (it is a marked length not a separate shape like a circle or a square). Fraction bars and tape diagrams (which are also called bar diagrams) are variations on the number line model.

Number line fractions are particularly good for representing improper fractions, for comparing fractions to whole numbers, and for thinking of fractions as a kind of number rather than a kind of shape.

On this number line, show the fractions 2/3, and 5/2:



CCSS: In grade 2 children work with simple fractions of geometric shapes:

> CCSS.Math.Content.2.G.A.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths…

In grade 3, the number line model for fractions is introduced, and is a topic of particular focus:

> CCSS.Math.Content.3.NF.A.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

Building on Unit Fractions: Understanding fractions as numbers (grades 3-5)

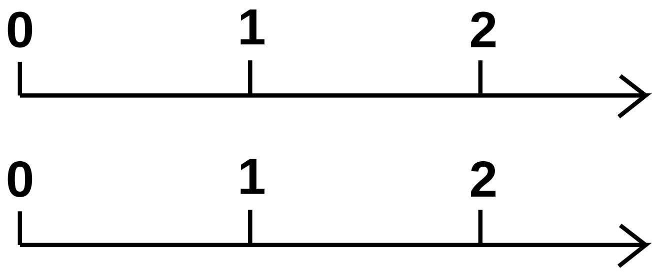
The most important idea for basic fraction knowledge in the common core standards is how to present fractions in terms of unit fractions, as described below:

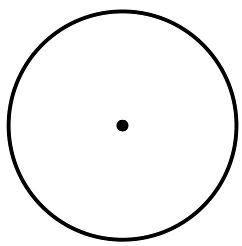
A **unit fraction** is a fraction whose numerator is 1. To show the unit fraction 1/b in any of our models, we start with a representation of 1 whole, and divide it into b equal parts. Each of those parts represents the fraction 1/b. This is always the way that these fractions are defined and presented, so only the name *unit fraction* should be new to you.

Fractions with a numerator greater than 1 are represented and explained as a sum of unit fractions, so 2/3 is 2 units of size 1/3, and 9/4 is 9 units of size ¼. This is slightly different from the most common way of presenting such fractions. The most common way of explaining 2/3 is to say that it is two out of 3 equal parts of a whole. That's a fine explanation for fractions that are less than 1 (proper fractions), but it leads to misunderstandings when children encounter fractions that are greater than 1 (improper fractions). This is the most fundamental change suggested for teaching and understanding fractions in the Common Core Standards compared to the way things have usually been done in the past, and it is a topic of emphasis in grade 3.

Practice: Using a number line and a circle model, show the steps (using unit fractions) to represent the fraction 7/4. Write a sentence explaining each step.

- The first step is to divide the whole unit into equal parts to show the size of the unit fraction (1/4)
- The second step is to draw out the right number of unit fractions to make the fraction you want to show (7/4)



Unit fraction 	
Repeated units:	

CCSS: Understanding fractions as repeated unit fractions is an explicit standard at third grade, and is again reiterated at fourth grade:

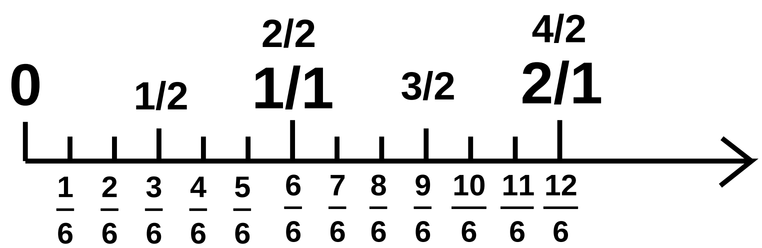
> CCSS.Math.Content.3.NF.A.1 Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.
>
> CCSS.Math.Content.4.NF.B.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

Fraction Equivalence and Comparing Fractions:

The teaching of fraction equivalence and fraction comparison is divided into two stages: a visual reasoning stage and a computational reasoning stage.

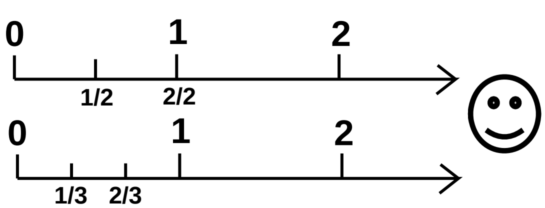
- In the *visual reasoning* stage, the main concepts are that the sizes of fractions can be compared using manipulatives and diagrams and by reasoning directly from the size of unit fractions.
- In the *computational* stage, students learn to generate equivalent fractions using multiplication and fractions are compared by finding a common denominator (usually) or a common numerator (occasionally).

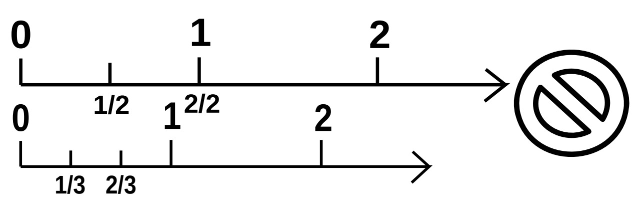
A **visual** approach to **equivalent fractions** is to say that two fractions are equivalent if they show the same part of the whole. In an area model (circles or rectangles), that means that the fractions cover the same area. In a number line (length) model, that means the fractions are at the same point on the number line or show the same length. Reasoning about this equivalence is often done by comparing physical models. There's a lot of room for error when comparing fractions with larger denominators, but the approach is helpful for developing visualization and estimation skills.



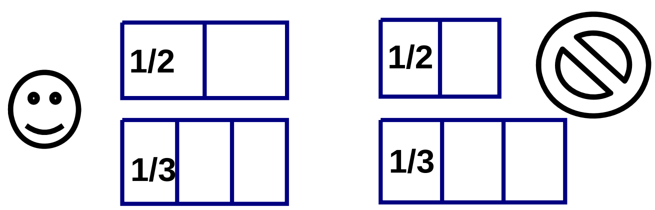
Another detail to notice is that the notation for writing a whole number as a fraction (2=2/1) is something that is now introduced in third grade along with putting fractions on a number line.

A visual approach to **comparing fractions** is to show that one fraction is larger than another if it shows a larger part of the same whole. In an area model, the area is larger, and in a length model the length is longer. The concept that *the size of the whole must be the same in order to compare fractions* is a key goal of instruction at this level. For example, to compare two fractions on number lines, the length of the whole unit 1 must be the same on both number lines:





The size of the rectangle that shows 1 must be the same for both fractions in a rectangular model:



CCSS: In grade 3, children should find equivalent fractions and compare fractions by reasoning about size using visual models and the meaning of fractions

> CCSS.Math.Content.3.NF.A.3 Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
>
> > CCSS.Math.Content.3.NF.A.3a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
> >
> > CCSS.Math.Content.3.NF.A.3b Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.
> >
> > CCSS.Math.Content.3.NF.A.3c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram*.

Reasoning with unit fractions can explain some comparisons without relying on a picture and also without computing common denominators. This kind of reasoning builds a good foundation for future fraction learning. The two most important examples of this are comparisons when either the denominators are already the same or the numerators are already the same in the two fractions.

- If the denominators are the same, then the fractions are composed of *unit fractions of the same size,* so the fraction with more units is larger.
- If the numerators are the same, then the fractions are composed of the same number of unit fractions, so we should compare the size of the unit fractions. Remember that unit fractions are made by dividing a whole into a given number of equal sized pieces. *If the whole is divided into more pieces, each unit fraction must be smaller*. The fraction with the larger denominator is composed of smaller unit fractions, so its size must be smaller.

Explaining comparisons using unit fractions is probably new to you. Practice with these pairs of fractions:

Explain:	Which is larger, 7/15 or 8/15?	Which is larger, 5/8 or 5/9?
• Are the unit fractions the same? • If the unit fractions are not the same, which is larger? Why? • Are there the same number of unit fractions? • How does the number of unit fractions and the size of unit fractions tell you which fraction shows the larger amount?		

Reasoning with unit fractions can also help you explain why some fractions are equivalent. If one of the denominators in a pair of fractions is a multiple of the other denominator, then you can use that fact to make a comparison to the larger unit fraction. For example, 12 is a multiple of 3, so we can compare 3rds to 12ths by saying: it takes 3 thirds to make a whole and it takes 12 twelfths to make a whole. If I glue together 4 twelfths, then 3 of those (3 sets of 4/12) is a whole, so 4/12 must be the same amount as 1/3.

Practice: Explain using the number of unit fractions of a whole, how to compare fifths and fifteenths (and tell how many fifteenths is the same amount as 1/5).

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CCSS: In grade 3, children should learn how to compare fractions with the same denominator and fractions with the same numerator:

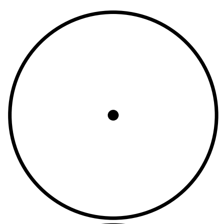
> CCSS.Math.Content.3.NF.A.3d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Learning to draw fractions whose denominator is a product can help you understand this process, and introduces ideas that are going to be important for understanding the computational way of comparing fractions. A good way to draw fractions whose numerator is a product is to split the whole into the fraction given by one factor, and then split each part into the number of parts given by the other factor. Try drawing these fractions whose denominator is a product:

To draw sixths: $\frac{1}{2\times 3}$

First draw in halves.

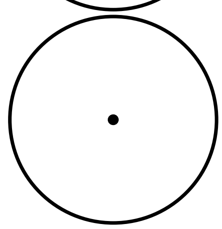
Then split each half into 3 equal parts



To draw twelfths: $\frac{1}{4\times 3}$

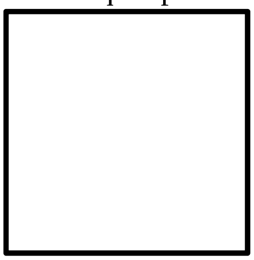
First draw in fourths

Then split each fourth into 3 equal parts

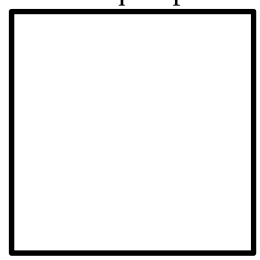


In a rectangular model, you can make subdividing lines vertically for one of the factors, and horizontally for the other factor:

To draw sixths: $\frac{1}{2\times 3}$
Draw 1 vertical line to make halves
Draw 2 horizontal lines to split each half into 3 equal parts



To draw twelfths: $\frac{1}{4\times 3}$
Draw 3 vertical lines to make fourths
Draw 2 horizontal lines to split each half into 3 equal parts



A **computational** approach to equivalent fractions is what you probably remember from learning fractions in elementary and middle school. The computational approach is/should be introduced *after* children have spent time developing concepts and intuition using manipulatives, drawings and reasoning from unit fraction size.

In the computational approach to equivalent fractions, we create equivalent fractions using multiplication and division. The computational approach is/should be introduced through pictures that children are familiar with using to represent fractions.

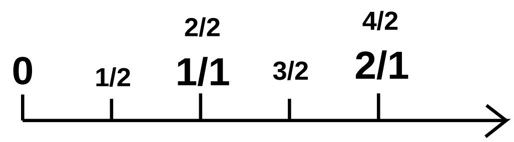
The split and split again strategy (split using the first factor and them split each part using the second factor) for drawing fractions with denominators that are a product of two numbers is a good introduction to the thinking that you need to build a computational strategy for finding common denominators.

- Unit fractions: A unit fraction $1/b$ is one of b equal parts of the whole. If you take that unit fraction and split it into n equal parts, then there are $n\cdot b$ equal parts of that size in the whole. Splitting the whole into b parts and each of those b parts into n parts gives b sets of n parts.   Thus, each part is a unit fraction $\frac{1}{nb}$.
- Composing several unit fractions: If the fraction is a/b then there are a parts of size $1/b$. If you split each of them into n equal parts, you will have $n\cdot a$ parts of size $\frac{1}{nb}$ so $\frac{a}{b}$ is the same amount as $\frac{na}{nb}$ (see below for examples of this reasoning using numbers).

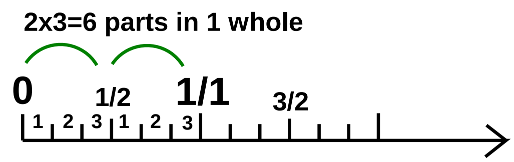
Explaining the process for specific examples: Notice that for each fraction below, I work with a model to create a fraction that I know is equivalent because it has *equal size*. I use the model and the fraction explanations to explain the computational strategy, so that *the numbers come from the diagram*, not vice versa.

Explaining equivalent fractions with a number line:

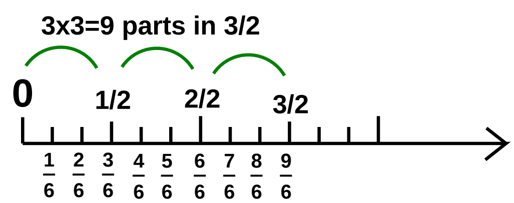
We're looking for a fraction equivalent to 3/2



Take each half and split it into 3 equal parts
That means each of the two halves in a whole are split into 3 equal parts, which makes $2 \cdot 3 = 6$ equal parts in a whole unit, so each of these smaller parts are $\frac{1}{2 \cdot 3} = \frac{1}{6}$
Now we know that there are 3 parts of size 1/6 in 1/2, so 1/2=3/6.



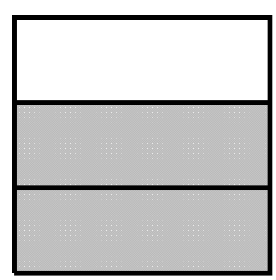
There are 3 parts of size 1/6 in each of the 3 halves. So that's $3 \cdot 3 = 9$ parts of size 1/6 in 3/2.
That means $\frac{3}{2} = \frac{3 \cdot 3}{3 \cdot 2} = \frac{9}{6}$



Notice that I'm writing out the answer using the factors: $3 \cdot 3$ and $3 \cdot 2$. That's me showing the computation strategy, which is: if you multiply the numerator and denominator by the same factor, you get an equivalent fraction.

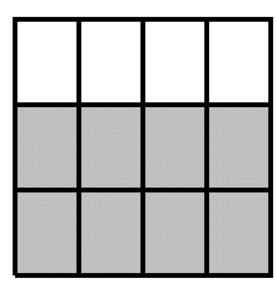
Explaining equivalent fractions in a rectangular model

We're looking for a fraction equivalent to 2/3. We represent 2/3 using horizontal subdivisions.



We split each of the 3 thirds into 4 equal parts by using a vertical subdivision

Each of the 2 shaded thirds were split into 4 smaller parts so there are 2×4 smaller shaded parts.



Each of the 3 thirds in a whole were split into 4 smaller parts, so there are 3×4 smaller parts in the whole, so each part has size $\frac{1}{12}$

Thus an equivalent fraction is $\frac{2 \times 4}{3 \times 4} = \frac{8}{12}$