Some important ideas:

1. If you have a set that includes the rational numbers, and is a subset of the complex numbers then to prove it is a field you only have to prove that it is closed under multiplication and addition and that it has multiplicative and additive inverses.
Things to think about:

- Why don't you have to prove that the set includes identities?
- Why don't you have to prove addition and multiplication are associative and commutative?
- What is the difference between being associative and being commutative?

2. If you have a field that includes another field (especially in the cases of fields F and G such that $\mathbb{Q}\subseteq F\subseteq G\subseteq\mathbb{C}$) then the larger field can be treated as a vector space where the scalars are the elements of the smaller field (we say G is a vector field over F). Sometimes these vector fields have a finite dimension and sometimes they have an infinite dimension. We have been focusing on vector fields that have a finite dimension. We write the dimension of the vector field [G:F].

3. Given a field (usually $\mathbb{Q}$, but I'm going to call it F in this description), and there is a number (α) that is not in F, but it is a solution of a polynomial with coefficients in F, then there is a vector field over F that has a finite number of basis vectors (one of which is α) that is also a field.
Things to think about:

- Is $\{a+b\sqrt{2}\mid a,b\in\mathbb{Q}\}$ a vector field? Is it a field? What is its dimension over $\mathbb{Q}$?
- Is $\{a+b\sqrt[3]{2}\mid a,b\in\mathbb{Q}\}$ a vector field? Is it a field? What is its dimension over $\mathbb{Q}$?
- Is $\{a+b\sqrt[3]{2}+c\sqrt[3]{2}^2\mid a,b,c\in\mathbb{Q}\}$ a vector field? Is it a field? What is its dimension over $\mathbb{Q}$?
- Is $\{a+b\sqrt[3]{2}+c\sqrt[3]{2}^2+d\sqrt[3]{2}^3\mid a,b,c,d\in\mathbb{Q}\}$ a vector field? Is it a field? What is its dimension over $\mathbb{Q}$?

Things to be able to do:

- Prove that $\{a+b\sqrt{2}\mid a,b\in\mathbb{Q}\}$ is a field
- Prove that $\{a+b\sqrt[3]{2}+c\sqrt[3]{2}^2\mid a,b,c\in\mathbb{Q}\}$ is a field
- Write $\mathbb{Q}(\alpha)$ as a vector field given that α is the root of an irreducible polynomial $x^5+3x^4+6x^2-9x+12$
- Given that α is the root of the irreducible polynomial x^4+5x^3-10 prove that $\{a+b\alpha+c\alpha^2+d\alpha^3\mid a,b,c,d\in\mathbb{Q}\}$ is closed under addition
- Given that α is the root of the irreducible polynomial x^4+5x^3-10 show that $3\alpha^6+2\alpha^5-\alpha^4+5\alpha^3\in\{a+b\alpha+c\alpha^2+d\alpha^3\mid a,b,c,d\in\mathbb{Q}\}$
- Given that α is the root of the irreducible polynomial x^4+5x^3-10 explain how to show that $\dfrac{1}{\alpha^2+3\alpha+2}$ is equal to a number in the set: $\{a+b\alpha+c\alpha^2+d\alpha^3\mid a,b,c,d\in\mathbb{Q}\}$

4. If you have polynomials with coefficients in the same field (think $\mathbb{Q}$), then you can add, subtract and multiply those polynomials, and you can do division with remainders. The coefficients always stay in the same field when you do that.

5. If you have two polynomials (p(x) and q(x)) with coefficients in the same field then you can use Euclid's algorithm to find the gcd of the two polynomials, and write an equation:
gcd = p(x)*f(x) +q(x)*g(x) where f(x) and g(x) are also polynomials with coefficients in the same field.
Things to think about:

- If p has degree n and it is irreducible, and q has degree m and m<n, is it possible for the gcd to have a degree greater than 0? Why not?
- If p is irreducible and α is a zero of p, can α be a zero of a polynomial that has a smaller degree than p? Why not?
- If p is irreducible over field F, and α is a zero of p, you can write the multiplicative inverse of 3+2 α as a polynomial in α (this means there is a polynomial q(x) where q(α) is the multiplicative inverse of 3+2 α). How does Euclid's algorithm make that possible?

Things to be able to do:

- Use Euclid's algorithm to find the gcd of two not-too-large polynomial as a "linear" combination of the polynomials. For example, for $p(x)=4x^3+8x+3$ and $q(x)=x^2-3x+2$, find polynomials *f* and *g* such that gcd(p,q) = p(x)*f(x) +q(x)*g(x)

6. Given a field (usually $\mathbb{Q}$, but I'm going to call it F in this description), and there is a number (α) that is not in F, but it is a solution of an irreducible polynomial of degree *n* with coefficients in F, then the basis of $F(\alpha)$ as a vector field with scalars in F is $\{1,\alpha,\alpha^2,...,\alpha^{n-1}\}$

Things to think about:

- Why don't we need any more basis vectors (why is this set enough)?
- Why do we need all of these basis vectors (why can't we get rid of any)?

Things to be able to do:

- Write a basis for a vector field for $\mathbb{Q}(\alpha)$ with scalars in $\mathbb{Q}$, where α is a root of the irreducible polynomial x^4+5x^3-10
- Tell the dimension $[\mathbb{Q}(\alpha):\mathbb{Q}]$ where α is a root of the irreducible polynomial x^4+5x^3-10

7. Understand the vector field representations of $\mathbb{Q}(\alpha,\beta)$, and the vector field dimension equation: If $\mathbb{Q}\subseteq F\subseteq G\subseteq H\subseteq\mathbb{C}$ and F, G, H are fields then $[H:F]=[H:G][G:F]$.

Things to be able to do:

- Write $\mathbb{Q}(\sqrt{5})$ as a vector field over $\mathbb{Q}$ and write $F(\sqrt{3})$ as a vector field over $F=\mathbb{Q}(\sqrt{5})$. Show how to get the basis for $\mathbb{Q}(\sqrt{5},\sqrt{3})$ as a vector field over from the previous representations.
- Given that c is a constructible number, explain why $[\mathbb{Q}(c):\mathbb{Q}]$ is a power of 2.
- Explain why $\cos(20°)$ is not constructible.
- Explain why $\sqrt[3]{5}$ is not constructible.