18. Explain why matrix multiplication is defined in the way that it is.

Matrix multiplication's main job (and best explanation) is to handle function composition of linear transformation matrices. We can see this with 2x2 matrices pretty easily.

The first thing to start with is the notion that we can use matrices to show what a linear transformation does. So if a linear transformation sends:

$\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}a\\b\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}c\\d\end{bmatrix}$ then it has to send $\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}xa+yc\\xb+yd\end{bmatrix}$

So a nice short-hand for that would be to use the matrix $\begin{bmatrix}a & c\\b & d\end{bmatrix}$

I'm going to call that function f

Now suppose we have another function, which I'll call g, and it sends:

$\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}m\\n\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}p\\q\end{bmatrix}$ then it has to send $\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}xm+yp\\xn+yq\end{bmatrix}$

So we'll use the matrix representation $\begin{bmatrix}m & p\\n & q\end{bmatrix}$

Now suppose we want the composition function $g\circ f$

It's going to send $\begin{bmatrix}1\\0\end{bmatrix}\to g\left(\begin{bmatrix}a\\b\end{bmatrix}\right)=\begin{bmatrix}ma+pb\\na+qb\end{bmatrix}$

and $\begin{bmatrix}0\\1\end{bmatrix}\to g\left(\begin{bmatrix}c\\d\end{bmatrix}\right)=\begin{bmatrix}mc+pd\\nc+qd\end{bmatrix}$ so we get the matrix form for $g\circ f$ to be $\begin{bmatrix}ma+pb & mc+pd\\na+qb & nc+qd\end{bmatrix}$

When I write out the matrices in order for the function composition I get

$$\begin{bmatrix}m & p\\n & q\end{bmatrix}\begin{bmatrix}a & c\\b & d\end{bmatrix}=\begin{bmatrix}ma+pb & mc+pd\\na+qb & nc+qd\end{bmatrix}$$

So the matrix multiplication rules tell you what you would get if you did one function and then the other.

19. The set $\{\vec{u} = <1,2>, \vec{v} = <-2,3>\}$ is a basis for $\mathbb{R}^2$.

a. How do we know it is a basis?

There are two linearly independent vectors in $\mathbb{R}^2$, so they must span the whole plane and be a basis.

b. Represent the vector <1,1> as a linear combination of $\vec{u}$ and $\vec{v}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5/7 \\ 0 & 1 & -1/7 \end{array}\right] \text{ so } <1,1> = \tfrac{5}{7}\vec{u} - \tfrac{1}{7}\vec{v}$$

c. Represent the vector <1,0> as a linear combination of $\vec{u}$ and $\vec{v}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3/7 \\ 0 & 1 & -2/7 \end{array}\right] \text{ so } <1,0> = \tfrac{3}{7}\vec{u} - \tfrac{2}{7}\vec{v}$$

d. Represent the vector <0,1> as a linear combination of $\vec{u}$ and $\vec{v}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/7 \\ 0 & 1 & 1/7 \end{array}\right] \text{ so } <0,1> = \tfrac{2}{7}\vec{u} + \tfrac{1}{7}\vec{v}$$

e. Write the matrix form of the linear transformation that maps <1,0> to $\vec{u}$ and <0,1> to $\vec{v}$.

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

f. Write the matrix form of the inverse transformation.

$$\begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{-2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

20. Write the matrix form of the linear transformation that maps <1,0,0> to <1,2,5>, and <0,1,0> to <-1, 1, 0>, and <0,0,1> to <2,1,1>.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

21. Write the matrix form of the linear transformation that maps <1,0> to <2,1> and <0,1> to <-4,-2>. Does this transformation have an inverse function? If so, find the invers. If not, why not?

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

This function does not have an inverse because it is not one-to-one (and it is not onto, and the are not linearly independent)

22. The linear transformation *f* maps <1,0> to $\vec{u}$ and maps <0,1> to $\vec{v}$.
Draw and label:

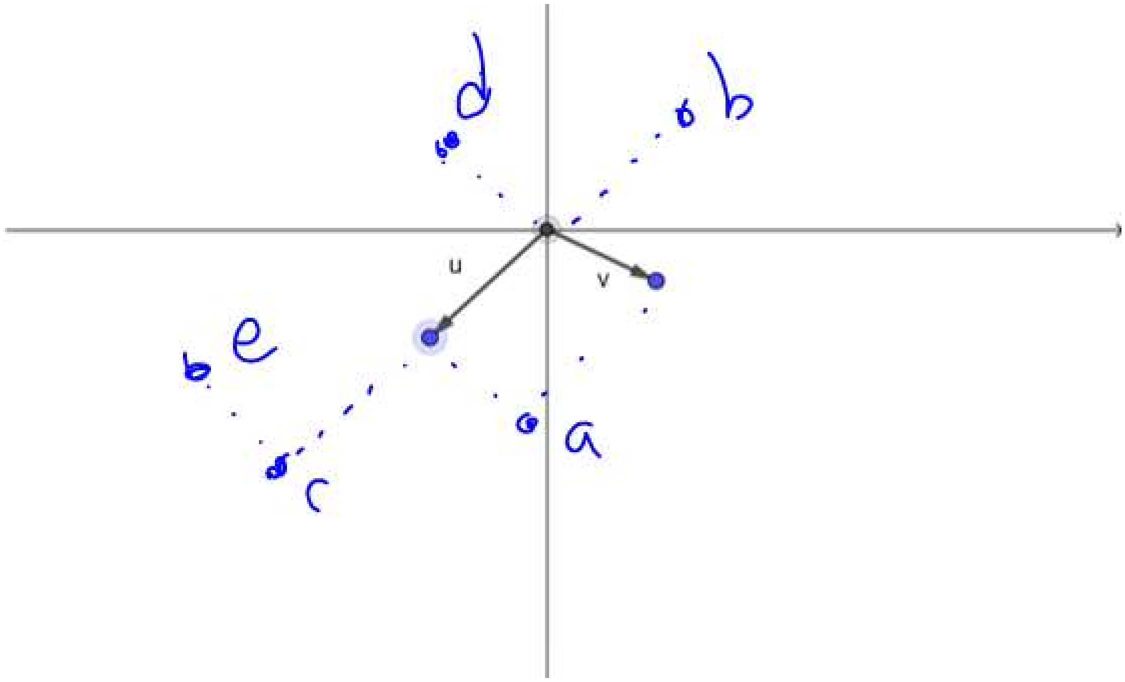
a. $f(<1,1>)$

b. $f(<-1,0>)$

c. $f(<2,0>)$

d. $f(<0,-1>)$

e. $f(<2,-1>)$



23. Explain in words how to interpret the dot product of two vectors in terms of projections.
The dot product of two vectors **u** and **v** is the length of the vector **u** projected onto the line defined by **v**, multiplied by the length of vector **v**.

24. Find the projection of <2,3> onto $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

$\left\langle \frac{54}{25}, \frac{72}{25} \right\rangle$

25. Find the projection of <-1,3> onto <1,2>
<1,2> (humph—not a very good problem)

26. Find the projection of <1, 2, 3> onto <1,0,1>
<2,0,2>