Practice problems for test:

$$A = \begin{pmatrix} -2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 8 & 17 \end{pmatrix}$$

1. Write the matrix multiplication you would do to perform the row operation where you switch Row 1 and Row 2 in matrix A (write the matrix, and show whether you would multiply it on the left or on the right of A)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 8 & 17 \end{pmatrix}$$

2. Write the matrix multiplication you would do to perform the row operation in matrix A where you replace Row 3 by 2 * Row 1 + Row 3 (write the matrix, and show whether you would multiply it on the left or on the right of A)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 8 & 17 \end{pmatrix}$$

3. Write the matrix multiplication you would do to perform the row operation where switch Column 1 and Column 2 in matrix A (write the matrix, and show whether you would multiply it on the left or on the right of A)

$$\begin{pmatrix} -2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 8 & 17 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Write the matrix multiplication you would do to multiply the first Column of A by 3.

$$\begin{pmatrix} -2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 8 & 17 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5. Show how to find $\det(A)$ by doing row operations/Gaussian Elimination.

$$\begin{pmatrix} -2 & -1 & 3 \\ 1 & 2 & 3 \\ 3 & 8 & 17 \end{pmatrix} \Rightarrow \begin{matrix} R_2 \\ R_1 \\ \\ \end{matrix}\begin{pmatrix} 1 & 2 & 3 \\ -2 & -1 & 3 \\ 3 & 8 & 17 \end{pmatrix} \Rightarrow \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + (-3)R_1 \end{matrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 9 \\ 0 & 2 & 8 \end{pmatrix} \Rightarrow R_3 * (1/3)\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \Rightarrow$$

$$*(-1) \qquad *(1) \qquad *(1/3)$$

$$\begin{matrix} \\ \\ R_3 + (-1)R_2 \end{matrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = 2 \text{ so}$$

$$*(1)$$

$$\det(A) * (-1) * (1/3) = 2$$

$$\det(A) = 2 * (-1) * (3) = -6$$

6. Find the inverse matrix of A (using the method of your choice)

$$A^{-1} = \begin{pmatrix} -10/6 & -41/6 & 3/2 \\ 4/3 & 43/6 & -3/2 \\ -1/3 & -13/6 & 1/2 \end{pmatrix}$$

7. Explain why you can (or prove that you can) do Gaussian elimination to find an inverse for a 2x2 matrix. You may use a specific example in your explanation, but your explanation should be generalizable to explain why it works for any 2x2 matrix.

I'd like you to be able to explain it in one of these two ways (so it's generalizable not just to any 2x2, but to any nxn matrix.

Explanation the first:

Suppose you have a matrix A that is a 2x2 matrix, and its row vectors are linearly independent. Then you can use row operations to convert it into the identity matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \ldots \to \ldots \to \ldots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Each of those row operations could be done by multiplying by a matrix on the left:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to E_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to E_2 E_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to E_3 E_2 E_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(traditionally these matrices are called elementary matrices, which is why I'm using E's to represent them.)

So, if it takes 4 row operations to get to the identity, you would have:

$$E_4 E_3 E_2 E_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If you multiply those row-operation elementary matrices by each other, instead of by A, we could say that

$E_4 E_3 E_2 E_1 = E$ and $EA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so we have found the inverse matrix of A: it's E. (!!!)

And $E_4 E_3 E_2 E_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E$

Now, remembering that each of the E's does a row operation, we can say:

$E_4 E_3 E_2 E_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E$ is what you get if you do all of the same row operations to $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So if you do row operations to A, and at the same time you *do the same row operations* to I, then when A gets down to the identity, I will have changed into E, which is the inverse of A.

Explanation the second:

Suppose you have a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that is a 2x2 matrix, and its column vectors are linearly independent.

That means that as a function $A\begin{bmatrix} x \\ y \end{bmatrix}$ maps (1,0) to the first column vector, and (0,1) to the second column vector.

Because the column vectors are linearly independent (and there are 2 of them) then they will be a basis for $\mathbb{R}^2$, so it's possible to write any vector as a linear combination of them, so it's possible to find scalars α, β such that $\alpha\begin{bmatrix} a \\ c \end{bmatrix} + \beta\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and also find scalars γ, δ such that $\gamma\begin{bmatrix} a \\ c \end{bmatrix} + \delta\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now if you were to make those into vectors and map them by $A\begin{bmatrix} x \\ y \end{bmatrix}$, you would get that:

$A\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix} = \alpha\begin{bmatrix} a \\ b \end{bmatrix} + \beta\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and similarly you would get $A\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

That means that

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so we've found the identity matrix, and it's $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ (!!!)

But wait, how did we find the scalars $\alpha, \beta, \gamma, \delta$?

Well we solved the systems of equations:

$\alpha\begin{bmatrix} a \\ c \end{bmatrix} + \beta\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \alpha a + \beta b = 1 \\ \alpha c + \beta d = 0 \end{matrix}$ and we probably did that by row operations on $\left[\begin{array}{cc|c} a & b & 1 \\ c & d & 0 \end{array}\right]$

and to find γ, δ, we probably did row operations on $\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 1 \end{array}\right]$

And we could combine those row operation steps, and that would find all of the constants we need for the inverse at the same time:

$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \alpha & \gamma \\ 0 & 1 & \beta & \delta \end{array}\right]$

Why I want you to know one of these two reasons for this way of finding inverse matrices:

These are good explanations because they scale well. Not only can you go from a single example to every 2x2, you can also go from 2x2 matrices to 3x3 matrices and even nxn matrices.

Explanation the first:

Suppose you have a matrix A that is a 3x3 matrix, and its row vectors are linearly independent. Then you can use row operations to convert it into the identity matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Each of those row operations could be done by multiplying by a matrix on the left:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow E_1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow E_2 E_1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \ldots \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(traditionally these matrices are called elementary matrices, which is why I'm using E's to represent them.)

So, if it takes n row operations to get to the identity, you would have:

$$E_n \ldots E_3 E_2 E_1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If you multiply those row-operation elementary matrices by each other, instead of by A, we could say that

$E_n \ldots E_3 E_2 E_1 = E$ and $EA = I$, so we have found the inverse matrix of A: it's E. (!!!)

And $E_n \ldots E_3 E_2 E_1 I = E$

Now, remembering that each of the E's does a row operation, we can say:

$E_n \ldots E_3 E_2 E_1 I = E$ is what you get if you do all of the same row operations to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So if you do row operations to A, and at the same time you do the same row operations to I, then when A gets down to the identity, I will have changed into E, which is the inverse of A.

Explanation the second:

Suppose you have a matrix $A=[v_1\ v_2\ v_3]$ that is a 3x3 matrix, and its column vectors are linearly independent.

That means that as a function $A\begin{bmatrix}x\\y\\z\end{bmatrix}$ maps $A\begin{bmatrix}1\\0\\0\end{bmatrix}=v_1$, $A\begin{bmatrix}0\\1\\0\end{bmatrix}=v_2$ and $A\begin{bmatrix}0\\0\\1\end{bmatrix}=v_3$

Because the column vectors are linearly independent (and there are 3 of them) then they will be a basis for $\mathbb{R}^3$, so it's possible to write any vector as a linear combination of them, so in particular, we can find scalars that solve the equations $a\,v_1+b\,v_2+c\,v_3=\hat{\imath}$, $d\,v_1+e\,v_2+f\,v_3=\hat{\jmath}$ and $g\,v_1+h\,v_2+i\,v_3=\hat{k}$

Now we can make column vectors out of each of those solutions, and they are the vectors that map to $\hat{\imath},\hat{\jmath},\hat{k}$ when you multiply on the left by A:

$$A\begin{bmatrix}a\\b\\c\end{bmatrix}=a\,v_1+b\,v_2+c\,v_3=\hat{\imath}=\begin{bmatrix}1\\0\\0\end{bmatrix}\text{, and similarly: }A\begin{bmatrix}d\\e\\f\end{bmatrix}=\hat{\jmath}=\begin{bmatrix}0\\1\\0\end{bmatrix}\text{ and }A\begin{bmatrix}g\\h\\i\end{bmatrix}=\hat{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

That means that if we put those column vectors we just found into a matrix and multiply by A, we get:

$$A\begin{bmatrix}a&d&g\\b&e&h\\c&f&i\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\text{, so we've found the identity matrix, and it's }\begin{bmatrix}a&d&g\\b&e&h\\c&f&i\end{bmatrix}\text{ (!!!)}$$

But wait, how could we find the scalars a,b,c,d,e,f,g,h,i?

Well we could find them by solving the systems of equations:

$a\,v_1+b\,v_2+c\,v_3=\hat{\imath}$ and we can solve that by doing row operations on the augmented matrix: $\begin{bmatrix}v_1 & v_2 & v_3 : \hat{\imath}\end{bmatrix}$

Similarly, $d\,v_1+e\,v_2+f\,v_3=\hat{\jmath}$ corresponds to the matrix $\begin{bmatrix}v_1 & v_2 & v_3 : \hat{\jmath}\end{bmatrix}$

And $g\,v_1+h\,v_2+i\,v_3=\hat{k}$ corresponds to the matrix $\begin{bmatrix}v_1 & v_2 & v_3 : \hat{k}\end{bmatrix}$

And we could combine those row operation steps, and that would find all of the constants we need for the inverse at the same time by getting the reduced row echelon form from

$[v_1\ v_2\ v_3 : \hat{\imath}\ \hat{\jmath}\ \hat{k}]$ which is also $[A{:}I]$

and when you row reduce, you get the scalars a,b,c,d,e,f,g,h,i so

$$[v_1\ v_2\ v_3 : \hat{\imath}\ \hat{\jmath}\ \hat{k}] \Rightarrow \left[\begin{array}{ccc|ccc}1&0&0&a&d&g\\0&1&0&b&e&h\\0&0&1&c&f&i\end{array}\right]$$