8. Give an example of a matrix where it would be faster to use Gaussian Elimination to find the determinant than it would be if you used expansion to find the determinant.

any matrix that's large-ish (4x4 or larger) and doesn't have very many 0s. For example:

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & -1 & 3 \\ 5 & 4 & 1 & 1 \\ 1 & 3 & -1 & -3 \end{bmatrix}$$

9. Give an example of a system of equations where you would prefer to use Cramer's Rule than use Gaussian Elimination (because it would be faster/easier). Explain your choice.

It seems to be just as fast and sometimes faster to use Cramer's rule for a 2-variable problem like:

$$\begin{aligned} 2x+3y&=5 \\ 3x-4y&=8 \end{aligned}$$

10. In the system of equations below, if you make a linear combination of the two equations, what kind of equation will you get? How will the graph of the linear combination be related to the graphs of the original equations? Explain how you know.

$$\begin{aligned} 3x+2y&=9 \\ 4x-y&=1 \end{aligned}$$

If you make a linear combination (that's not just multiplying by 0), then you will get another linear equation (equation of a line) that passes through the point of intersection of the first two lines.

When we make the new equation, we'll be adding:

$a(3x+2y)+b(4x-y)=9a+b$ which simplifies to be:

$(3a+4b)x+(2a-b)y=(9a+b)$ which is an equation of a line.

If both equations have a common solution/common point (x_0, y_0), then

$\begin{aligned} 3x_0+2y_0&=9 \\ 4x_0-y_0&=1 \end{aligned}$ so that means $\begin{aligned} (3x_0+2y_0)a&=9a \\ (4x_0-y_0)b&=1b \end{aligned}$ which means it has to be true that

$$a(3x_0+2y_0)+b(4x_0-y_0)=9a+b$$

so (x_0, y_0) is also a solution for the linear combination linear equation.

11. Prove that you can get every line through (1,2) by making linear combinations of

$2x + y = 4$

$x - 3y = -5$

The Linear combinations are: $a(2x + y) + b(x - 3y) = 4a - 5b$

which simplifies to $(2a + b)x + (a - 3b)y = 4a - 5b$

And solving for y, we get:

$y = \frac{-(2a+b)}{a-3b}x + \frac{4a-5b}{a-3b}$ which is a line with slope $m = \frac{-(2a+b)}{a-3b}$

We know that (1,2) is a point on all of the lines because it is on both of the original 2 lines. We could also check by substituting x=1:

$$y = \frac{-(2a+b)}{a-3b} \cdot 1 + \frac{4a-5b}{a-3b} = \frac{2a-6b}{a-3b} = 2$$

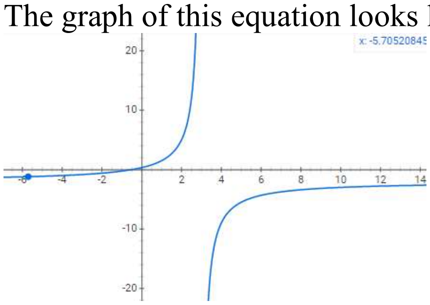
To understand the slope equation, it helps to make a simplifying assumption like one of these:

What values of m are possible if b=1?

$$m = \frac{-(2a+1)}{a-3} = \frac{-2a-1}{a-3} = \frac{(-2a+6)-7}{a-3}$$

$$= -2 + \frac{-7}{a-3}$$

The graph of this equation looks like this:



With a horizontal asymptote at m=-2, so we can get all real values of m except -2 when b=1 just by changing a.

Can we solve for m=-2 when a=1?

$$\frac{-(2+b)}{1-3b} = -2 \;\; =>$$

$$-2 - b = -2(1-3b) \;=>\; -2 - b = -2 + 6b \;=>\; b = 0$$

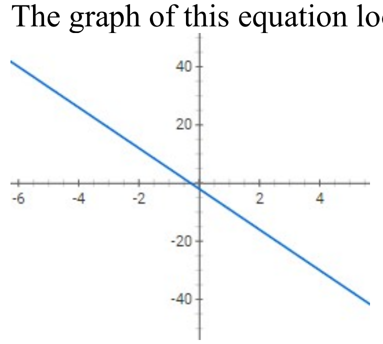
So if a=1 and b=0 then $m = \frac{-(2+0)}{1-3\cdot 0} = -2$

What values of m are possible if a-3b=1?

Then $a = 1 + 3b$

So $m = \frac{-(2(1+3b)+b)}{1} = -2 - 7b$

The graph of this equation looks like this:



So we can get any real value for m by choosing an appropriate value for b.

What about a vertical line? We would expect to get that when m is undefined, so when $a - 3b = 0$

Let's see what happens when b=1 and a=3. The slope-intercept form doesn't work for that, so we need to go back to the general form:

$(2a + b)x + (a - 3b)y = 4a - 5b \;=>\; (2\cdot 3 + 1)x + (3 - 3\cdot 1)y = 4\cdot 3 - 5\cdot 1 \;=>\; 7x + 0y = 7 \;=>\; x = 1$ which is the vertical line through (1,2)

12. Here are some sets of vectors in $\mathbb{R}^3$. For each set tell whether it is independent, and whether it spans $\mathbb{R}^3$.

a. {(1, 1, 0), (2, 2, 0)}
b. {(1, 1, 0), (0, 1, 1)}
c. {(1, 1, 0), (0, 1, 1), (1, 2, 1), (1, 0, -1)}
d. {(1, 1, 0), (1, 2, 1), (0, 1, 1), (0, 1, 0)}
e. {(1, 1, 0), (1, 2, 1), (0, 1, 0)}
f. {(1, 1, 0), (1, 2, 1), (0, 1, 1)}

13. Here are some sets of vectors in $\mathbb{R}^4$. For each set tell whether it is independent, and whether it spans $\mathbb{R}^4$.

a. {(1, 2, 3, 4), (1, 1, 0, 0), (1, 0, 0, 1)}
b. {(1, 2, 3, 4), (1, -1, 3, 5), (1, 5, 3, 3)}
c. {(1, 2, 3, 4), (1, 0, 1, 2), (2, 1, 1, 0), (2, 3, 3, 2), (3, 4, 3, 0)}
d. {(1, 2, 3, 4), (1, 0, 1, 2), (2, 1, 1, 0), (1, 2, 0, 1), (1, 1, 0, 0)}
e. {(1, 2, 3, 4), (1, 0, 1, 2), (2, 1, 1, 0), (1, 2, 0, 1)}
f. {(1, 2, 3, 4), (5, 4, 2, 1), (2, 1, 1, 0), (1, 2, 0, 1)}

Note 1: you can put these vectors into a matrix in two ways: as rows or as columns. Then you can row reduce (or get your calculator to row reduce): what does the reduced form tell you? Do you prefer working with the vectors as rows or as columns?

If you put the vectors in as rows, then if there is a row of 0s, you know that it was a dependent system, and if there is no row of 0s, the vectors were independent. The number of non-zero rows is the number of independent vectors in the set.

If you put the vectors in as columns, then if there is a row that has more than one non-zero number in it, that makes the system dependent. You can use the reduced rows to find an equation that represents one of the column vectors in terms of the other vectors (so you can tell not only how many independent vectors there are, but which ones you could drop from the list without changing the spanning set).

If there are n independent vectors, then it spans an n-dimensional flat space (line, plane, hyperplane)

Note 2: I on purpose gave you easier examples in #12 (ones you can probably figure out without row reducing). Can you use those as examples to help you make sense of #13?

14. The reduced row echelon form of a matrix B is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

a. Each of the 5 rows of the matrix are vectors in $\mathbb{R}^6$. Are these vectors linearly independent? How do you know?

The vectors are linearly independent (because there are no 0-rows)

b. Do these vectors span $\mathbb{R}^6$? How do you know?

There are only 5 vectors, so they span a 5-dimensional hyperplane, but not all of $\mathbb{R}^6$.

c. Each of the 6 columns of the matrix are vectors in $\mathbb{R}^5$. Are these vectors linearly independent? How do you know?

No, these are not linearly independent. You can tell because the rows have more than one non-zero term, so you could come up with a linear combination of the last one in terms of the first 5.

d. Do these vectors span $\mathbb{R}^5$? How do you know?

Yes, they do. There are 5 independent columns, so they span $\mathbb{R}^5$.

I can tell that there are 5 independent columns because there are 5 columns left where the column has just one 1, and the rest are 0.

15. The reduced row echelon form of a matrix C is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a. Each of the 5 rows of the matrix are vectors in $\mathbb{R}^6$. Are these vectors linearly independent? How do you know?

These vectors are linearly dependent. (one row of 0s)

b. Do these vectors span $\mathbb{R}^6$? How do you know?

No, there are only 4 rows left, so they span a 4-dimensional hyperplane.

c. Each of the 6 columns of the matrix are vectors in $\mathbb{R}^5$. Are these vectors linearly independent? How do you know?

No, they are dependent. There are two columns where there are more than one no-zero value.

d. Do these vectors span $\mathbb{R}^5$? How do you know?

No, there are only 4 independent vectors, so they span a 4-dimensional hyperplane.

16. The reduced row echelon form of a matrix D is:

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a. The four column vectors of D do not span $\mathbb{R}^4$. What about the reduced form of the matrix tells you that?
There are two rows with more than one non-zero term.

b. The four column vectors of D are not linearly independent. If you were going to drop one of the column vectors from the list because it is "extra" (dependent on one or more of the others) which one or ones could you eliminate? Why?
I would probably drop the fourth one, but you could also drop the first or the third vector. The second vector is independent of the other 3 (only one non-zero term in the second row), so you can't drop it.

c. Does this reduced row echelon form say that the first vector is 3 times the fourth vector? If not, what does it say?
No: it says that to get a linear combination where $a\,v_1 + b\,v_2 + c\,v_3 + d\,v_4 = 0$, you would need to have a+3d=0.
You would also have to have c+2d=0, so all three of those coefficients would be non-zero.
An example that would work would be $-3 \cdot v_1 + 0 \cdot v_2 + (-2) \cdot v_3 + 1 \cdot v_4 = 0$

17. Tell a basis for $\mathbb{R}^3$ that does not include any of $\hat{i}$, $\hat{j}$ or $\hat{k}$.

{(1,2,0), (0, 1, 1), (1,1,1)}