Solving cubic equations using Cardan's method:

$(y+d)^2 = y^2 + 2dy + d^2$

3b Step 1: get rid of $x^2$

What we're using:

$(y-d)^3 = y^3 - 3dy^2 + 3d^2y - d^3$ and $(y+d)^3 = y^3 + 3dy^2 + 3d^2y + d^3$

$x^3 - 6x^2 + 15x - 18 = 0$

$x = y - (-6/3) = y + 2$

$(y+2)^3 - 6(y+2)^2 + 15(y+2) - 18 = 0$

$y^3 + 3\cdot 2y^2 + 3\cdot 2^2 y + 2^3 - 6(y^2 + 2\cdot 2y + 2^2) + 15y + 30 - 18 = 0$

$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 15y + 12 = 0$

$y^3 + 3y - 4 = 0$

check your work

Step 2: $y = a - b$

What we're using:

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a-b)^3 + 3a^2b - 3ab^2 = a^3 - b^3$

$(a-b)^3 + 3ab(a-b) = a^3 - b^3$

We want $y = a - b$

So then

$y^3 + 3y - 4 = 0$

$$\begin{array}{ccccc} y^3 & +3 & y & = & 4 \\ (a-b)^3 & +3ab & (a-b) & = & a^3 - b^3 \end{array}$$

Which gives two equations in two variables:

$$\begin{array}{ll} ab = 1 & a^3 - b^3 = 4 \\ (ab)^3 = 1^3 & (a^3 - b^3)^2 = 4^2 \\ a^3b^3 = 1 & a^6 - 2a^3b^3 + b^6 = 16 \\ 4a^3b^3 = 4 & + \quad 4a^3b^3 = 4 \\ & a^6 + 2a^3b^3 + b^6 = 20 \\ & (a^3 + b^3)^2 = 20 \\ & a^3 + b^3 = \sqrt{20} \\ & a^3 + b^3 = 2\sqrt{5} \end{array}$$

$x - y = 4$

$x + y = 2\sqrt{5}$

Which turns into a different two equations in two variables:

$$\begin{array}{ll} a^3 - b^3 = 4 \quad \times(-1) & -a^3 + b^3 = -4 \\ + \; a^3 + b^3 = 2\sqrt{5} & + \; a^3 + b^3 = 2\sqrt{5} \\ 2a^3 = 4 + 2\sqrt{5} & 2b^3 = 2\sqrt{5} - 4 \\ a^3 = 2 + \sqrt{5} & b^3 = \sqrt{5} - 2 \\ a = \sqrt[3]{2+\sqrt{5}} & b = \sqrt[3]{\sqrt{5}-2} \end{array}$$

From which we go back to $y$:

$$y = a - b = \sqrt[3]{2+\sqrt{5}} - \sqrt[3]{\sqrt{5}-2}$$

and then back to $x$:

$$x = y + 2 = 2 + \sqrt[3]{2+\sqrt{5}} - \sqrt[3]{\sqrt{5}-2}$$

If you evaluate that on a calculator, you will get 3 (it’s tricky to prove algebraically that it’s 3!). This strategy always gets the right answer, but it’s not always in a friendly form.

3c: $x^3 - 6x^2 + 27x - 58 = 0$ $\quad x = (y+2)$

3b Step 1: get rid of $x^2$

What we're using:

$(y-d)^3 = y^3 - 3dy^2 + 3d^2y - d^3$ and $(y+d)^3 = y^3 + 3dy^2 + 3d^2y + d^3$

$x^3 - 6x^2 + 15x - 18 = 0$

$x = y - (-6/3) = y + 2$

$(y+2)^3 - 6(y+2)^2 + 27(y+2) - 58 = 0$

$y^3 + 3\cdot 2y^2 + 3\cdot 2^2 y + 2^3 - 6(y^2 + 2\cdot 2y + 2^2) + 27y + 54 - 58 = 0$

$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 27y - 4 = 0$

$y^3 + 15y - 20 = 0$

*check your work with the formulas in the book at this point!*

Step 2: $y = a - b$

What we're using:

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a-b)^3 + 3a^2b - 3ab^2 = a^3 - b^3$

$(a-b)^3 + 3ab(a-b) = a^3 - b^3$

We want $y = a - b$

So then

$y^3 + 15y - 20 = 0$

$$\begin{array}{cccccc} y^3 & +15 & y & = & 20 \\ (a-b)^3 & +3ab & (a-b) & = & a^3 - b^3 \end{array}$$

$(a^3 - b^3)^2 = \mathrm{I}$

$(a^3 + b^3)^2 = \mathrm{II}$

Which gives two equations in two variables:

| | |
|---|---|
| $ab = 5$ | $a^3 - b^3 = 20$ |
| $(ab)^3 = 5^3$ | $(a^3 - b^3)^2 = 400$ |
| $a^3b^3 = 125$ | $a^6 - 2a^3b^3 + b^6 = 400$ |
| $4a^3b^3 = 500$ | $+ \quad 4a^3b^3 = 500$ |
| | $a^6 + 2a^3b^3 + b^6 = 900$ |
| | $(a^3 + b^3)^2 = 900$ |
| | $a^3 + b^3 = 30$ |

Which turns into a different two equations in two variables:

$$a^3 - b^3 = 20$$
$$+\ a^3 + b^3 = 30$$
$$2a^3 = 50$$
$$a^3 = 25$$
$$a = \sqrt[3]{25}$$

$$-a^3 + b^3 = -20$$
$$+\ a^3 + b^3 = 30$$
$$2b^3 = 10$$
$$b^3 = 5$$
$$b = \sqrt[3]{5}$$

From which we go back to $y$:

$$y = a - b = \sqrt[3]{25} - \sqrt[3]{5}$$

and then back to $x$:

$$x = y + 2 = 2 + \sqrt[3]{25} - \sqrt[3]{5}$$

This is an unusually nice answer to a cubic in my experience.  Your homework comes out of the book, so you can expect to also have unusually nice answers like this.