Geometry Axioms and Theorems

Definition: The **plane** is a set of points that satisfy the axioms below. We will sometimes write E^2 to denote the plane.

Axiom 1: Given any two points, A and B in the plane, there is one and only one line $\overleftrightarrow{AB}$ that contains both points, one and only one segment $\overline{AB}$ that has those points as endpoints, and one and only one ray $\overrightarrow{AB}$ that starts at the first point and continues through the second.

Theorem 1: Distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in at most one point.

Axiom 2: Between any two points, A and B, we can measure the distance AB, which is a non-negative real number. The distance between any two points is also the length of the segment between the points $\overline{AB}$. If C is a point on the segment $\overline{AB}$, then the lengths satisfy $AC + CB = AB$ and if D is a point that is not on the segment $\overline{AB}$, then $AD + DB > AB$

Definition: Given a (center) point A and a (radius) distance r, which is a non-negative real number, then the **circle** with center A and radius r is defined to be:
$\odot(A,r) = \{P \mid P \text{ is a point in the plane, and the distance } AP = r\}$

Axiom 3: Lines and circles are continuous

Theorem 2: If P is the endpoint of the ray $\overrightarrow{PA}$, and r is a distance (non-negative real number), then there is one and only one point B that is at the intersection of the circle $\odot(P,r)$ and the ray $\overrightarrow{PA}$ (so that B is on ray $\overrightarrow{PA}$, and the distance $PB = r$

Definition: Given a (center) point A and a (radius) distance $r \in [0,\infty)$, the **circle** with center A and radius r is defined to be $\odot(A,r) = \{P \mid P \text{ is a point in the plane, and the distance } AP = r\}$

Axiom 3: For any non-collinear sets of points A, B, C and distinct points D, E and a specified side of $\overleftrightarrow{DE}$, there exists a unique isometry $f : E^2 \to E^2$ such that $f(A) = D$ and $f(B) \in \overrightarrow{DE}$ and $f(C)$ lies on the specified side of $\overleftrightarrow{DE}$. (Unique means there is one and only one such isometry)

Axiom 4: For any three points in the plane, *A, B, C*, there is an angle $\angle ABC$, and we can measure the angle in degrees.

Definition: Two line segments are congruent if they have the same length, two angles are congruent if they have the same angle measurement, and two triangles are congruent if the sides and angles are congruent in a matching order.

Definition: Two triangles ΔABC and ΔDEF can be assigned a **correspondence**, which means that the sides and angles that are named in the same order are said to be corresponding sides and angles: $\overline{AB}$ corresponds to $\overline{DE}$, $\angle BAC$ corresponds to $\angle EDF$, etc.

SAS axiom (5): If two sides of one triangle are congruent to corresponding two sides of another triangle, and the angles between those sides are congruent, then the triangles are congruent.

ASA Theorem (3): If two angle of one triangle are congruent to corresponding two angles of another triangle, and the sides between those angles are congruent, then the triangles are congruent.

The Isosceles Triangle Theorem

a) If two sides in a triangle are congruent, then the angles opposite the sides are also congruent.

b) If two angles in a triangle are congruent, then the sides opposite the angles are also congruent.

SSS Theorem (4): If all three sides of a triangle are congruent to the corresponding three sides of another triangle, then the triangles are congruent.

AAS Theorem (5): If two angles of a triangle are congruent to the corresponding two angles of another triangle, and if a side of one triangle (that is not between the two angles) is congruent to the corresponding side of the other triangle, then the triangles are congruent.

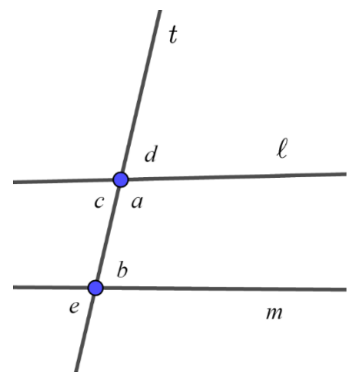
HL Theorem (5.5): If the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent.

CPCTC: Corresponding parts of congruent triangles are congruent. This is the definition of "congruent triangles".

VA Theorem (6): If two lines intersect, then they make vertically opposite angles that are congruent to each other.

Definition/example: Given a pair of lines (in the diagram ℓ, m), intersected at distinct points by a transversal (*t* in the diagram), then, using the angles named in the diagram:

- a and b are interior angles on the same side of the transversal
- b and c are alternate interior angles
- b and d are corresponding angles
- d and e are alternate exterior angles



Definition: Two lines in the plane are parallel if they do not intersect. Two segments are parallel if the corresponding infinite lines are parallel.

CA Axiom (6): Given two lines are cut by a transversal, the corresponding angles are congruent if and only if the lines are parallel.

AIA Theorem (7): Given two lines are cut by a transversal, the alternate interior angles and congruent if and only if the lines are parallel.

AEA Theorem (8): Given two lines are cut by a transversal, the alternate exterior angles are congruent if and only if the lines are parallel.

Angles between parallels theorem (8.5): Given two lines cut by a transversal, the angles interior to the two lines on the same side of the transversal are supplementary (add to 180°) if and only if the lines are parallel. In the diagram, these would be angles *a* and *b*.

Triangle Sum Theorem (9): The sum of the angles in a triangle is 180°.

Exterior to a triangle theorem (9.1): The measure of an angle exterior to a triangle is equal to the sum of the measures of the two opposite and interior angles.

Angles in a quadrilateral theorem (9.2): The sum of angles in a quadrilateral is 160°

Definition: A **rectangle** is a quadrilateral (4-sided polygon) where all the angles are right angles. A **square** is a rectangle whose sides are all equal.

Definition: A **parallelogram** is a quadrilateral where the opposite sides are parallel.

Definition: A **rhombus** is a quadrilateral all of whose sides are congruent.

Definition: A **kite** is a quadrilateral where a pair of adjacent sides are congruent to each other, and the other two sides are also adjacent to and congruent to each other.

Definition: The **area** of a square with side lengths 1 unit is 1 square unit. Areas are measured in square units, where congruent shapes have the same area, and if a shape with area is cut into two smaller shapes, the sum of the areas of the smaller shape is the area of the larger shape

Opposite sides of a parallelogram theorem (9.5): The opposite sides of a parallelogram are congruent.

Opposite sides of a rectangle theorem (9.6): The opposite sides of a rectangle are parallel and congruent.

Area of a Rectangle Theorem (10): A rectangle with length a units and width b units has area ab square units.

Area of a Parallelogram Theorem (11): A parallelogram with one side denoted as a base with length b, and a height h which is the distance between the two parallel lines has area bh

Area of a Triangle Theorem (11): In any triangle, if you choose a side of the triangle to be the base, and call its length b, then the segment from the opposite vertex to the line that includes the base which is perpendicular to the line that includes the base is called the height h, and the area of the triangle is $\frac{1}{2}bh$

Pythagorean Theorem (12): In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

Definition: Two triangles (or polygons) are similar if their corresponding angles are equal and their corresponding side lengths are proportional.

AA Similarity (13): Given two triangles, if two of the angles in one triangle are congruent to the corresponding two angles in the other triangle, then the triangles are similar.

SAS Similarity (14): Given two triangles, if two of the sides in one triangle are proportional to the corresponding sides in the other triangle, and if the angles between those two sides are equal, then the triangles are similar.

Chord bisected by radius theorem (15): In a circle, given a chord (that is not a diameter) that is intersected by a radius (line through the center of the circle), then the radius bisects the chord (intersects it at its midpoint) if and only if the radius and the chord are perpendicular.

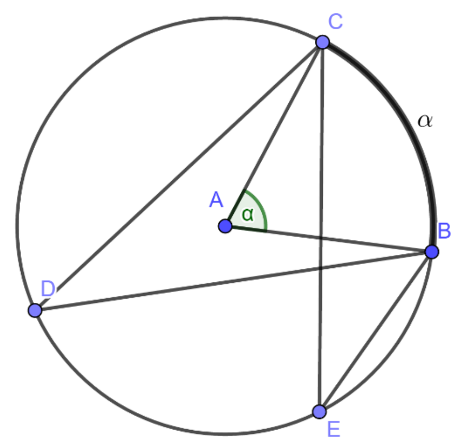
Side lengths within a triangle theorem (16): Given a triangle with two sides of lengths a and b, which are opposite two angles A and B, then $a>b$ if and only if $A>B$.

Perpendicular to a line theorem (17): Given a line ℓ, and a point P that is not on the line, it is possible to construct a line m that includes P and is perpendicular to ℓ.

Perpendicular length theorem (17.5): Given a line ℓ, and a point P that is not on the line, and segment $\overline{PQ}$ such that $Q \in \ell$ and $\overline{PQ} \perp \ell$, then $\overline{PQ}$ is shorter than any other segment from P to a point on the line.

Definition: An infinite line is tangent to a circle if it intersects the circle once and only once.

Tangent to a circle theorem (18): Given a circle, with point P being a point on the circle, then a line through point P is tangent to the circle (intersects the circle once and only once) if and only if it is perpendicular to the radius of the circle to the point P.



Definition/example: In a circle, the angle $\alpha = \angle BAC$, where the vertex is the center of the circle, is a **central angle.** An arc $\widehat{BC}$ on a circle is sometimes called an **arc angle**, and when it is called an arc angle, then it's angle measurement is the same as the angle measurement of the central angle that it corresponds to. The angles $\angle BDC$ and $\angle BEC$, where the vertex is a point on the circle is called an **inscribed angle**. We sometimes say that the angles $\angle BAC$, $\angle BDC$, and $\angle BEC$ are all **subtended** by the arc $\widehat{BC}$ when we want to talk about the relationship between the arc and the angle(s).

Inscribed angle theorem (19): The measure of an inscribed angle is half of the measure of the central angle that subtends the same arc.

Angle in semi-circle theorem (20): An angle inscribed in (and subtended by) a semi-circle is a right angle.

Inscribed quadrilateral theorem (21): If a quadrilateral is inscribed in a circle (all of its vertices are on the circle), then the opposite angles in the quadrilateral are supplementary (add to 180°).

Power of a point theorem (22): If C is a circle, and P is a point that is not on the circle, then there is a number $pow(P,C)$ such that if l is any line containing P, and A and B are the intersections of l with C, then $(PA)(PB) = pow(P,C)$