Circumcenter theorem: Proving that perpendicular bisectors of the sides of a triangle are concurrent, and are the center of a circumscribed circle for the triangle:

Using lemma 1: Given segment $\overline{AB}$, a point C lies on the perpendicular bisector of $\overline{AB}$ if and only if it is equidistant from points *A* and *B*.

Proof: Given a triangle ΔABC, let lines *l*, *m* and *n* be the perpendicular bisectors of sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

Let $D = l \cap m$

Then $D \in l$

by lemma 1, $DA = DB$

Also, $D \in m$

by lemma 1, $DB = DC$

by the transitive property, because $DA = DB$ and $DB = DC$, therefore $DA = DC$

by lemma 1, *D* lies on the perpendicular bisector of $\overline{AC}$, so $D \in n$

Thus, *D* lies on all three perpendicular bisectors, so they are concurrent.

A circle with center *D* and radius with length DA will go through all three vertices of the triangle because $DA = DB = DC$, so the circle circumscribes the triangle.

Incenter theorem: Proving that angle bisectors of the angles of a triangle are concurrent, and are the center of an inscribed circle for the triangle:

Using lemma 2: Given angle $\angle ABC$, a point D lies on the angle bisector of $\angle ABC$ if and only if the distance from D to $\overleftrightarrow{AB}$ is equal to the distance from D to $\overleftrightarrow{BC}$

Proof:

First name the triangles, and name the angle bisectors.

Next name the intersection of two of the angle bisectors

State that the intersection point is on one particular angle bisector. Construct and name the perpendicular lines to the sides of the angle so you can use lemma 2 to say two lengths are equal

State that the intersection point is on the other angle bisector. Construct and name the perpendicular line to the third side of the triangle, so you can use lemma 2 to say 2 lengths are equal

Use the transitive property to write down a third equation

Use lemma 2 to say that your intersection point lies on the third angle bisector.

Conclude that the angle bisectors are concurrent

Construct a circle whose center is the point where the angle bisectors are concurrent, and whose radius is one of the other lengths you constructed.

Explain why your circle is tangent to the side of the triangle

Explain why your circle is tangent goes through two other points of the triangle, and why it is tangent to the other two sides of the triangle

Centroid theorem: Proving that medians of the sides of a triangle are concurrent (Centroid).

Using lemma 4b: Given triangle ΔABC, with *M* being the midpoint of side $\overline{AC}$ and *N* being the midpoint of side $\overline{BC}$, and $P = \overline{BM} \cap \overline{AN}$ then $(PM) = \frac{1}{3}(BM)$

Proof: Given triangle ΔABC, with *M* being the midpoint of side $\overline{AC}$ and *N* being the midpoint of side $\overline{BC}$ and *Q* being the midpoint of side $\overline{AB}$.

I have named the triangle and its medians

Then the medians of ΔABC are $\overline{AN}$, $\overline{BM}$ and $\overline{CQ}$.

I have intersected two medians and written down the properties of that intersection using lemma 4b

Let $P = \overline{BM} \cap \overline{AN}$, then by lemma 4b, $P \in \overline{BM}$ and $(PM) = \frac{1}{3}(BM)$

Also by lemma 4b, because $P \in \overline{AN} \cap \overline{BM}$, $P \in \overline{AN}$ and $(PN) = \frac{1}{3}(AN)$

Let $P' = \overline{BM} \cap \overline{CQ}$, then by lemma 4b,

Next, I have named the intersection of a different pair of medians. Write down the properties of that intersection using lemma 4b

Find the properties of *P* that is the same as a properties of P' and use them to prove that $P = P'$

Conclude that the medians are concurrent at point *P*.

Note: This point of concurrency is called the Centroid. In lemmas 4a and 4b, we used the fact that the medians divide the area of the triangle into equal halves. It's more difficult, but possible to prove that any line that contains the centroid will divide the area of the triangle into equal halves. It's that property (the areas/masses on either side of a line through the centroid are equal) that makes the centroid the balance point or center of gravity for the triangle (assuming that the triangle has evenly distributed mass/density)

Orthocenter theorem: Proving that (extended) altitudes of the sides of a triangle are concurrent (Orthocenter).

Using lemma 3: Given triangle ΔABC, if you construct lines parallel to each side going through the opposite vertex, that will create a large triangle that is split into four smaller triangles, one of which is ΔABC, and each of the smaller triangles are congruent to ΔABC.

Name the triangle and its extended altitudes. By extended, we mean that these are infinite lines, not just segments
Construct the lines described in the lemma, and name the intersection points so the larger triangle is named
Prove, using the congruent triangles from the lemma, that A, B and C (vertices of the original triangle) are the midpoints of the sides of the larger triangle.
Prove, using the parallel sides from the construction of the triangle, that the altitude at A is perpendicular to the side of the larger triangle that contains A.
Note that the altitudes of ΔABC are the perpendicular bisectors of the larger triangle.
Use the Circumcenter Theorem to conclude that the altitudes must be concurrent in a single point.