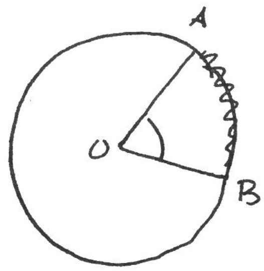
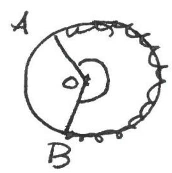
Geometry summary Feb 18-20



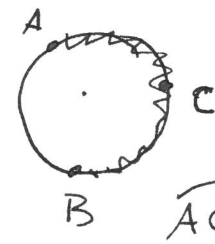
the measurement of an arc-angle $\overset{\frown}{AB}$ is defined to be the measure of the central angle $\angle AOB$.

This theorem is usually stated in terms of arc-angles, because the central angle is confusing when the angle is $>180°$ (often angles are defined to always be $\leq 180°$)

For example:

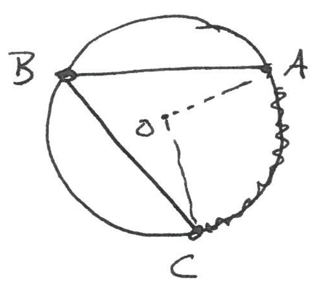


You can make the arc-angle more clear by adding a point



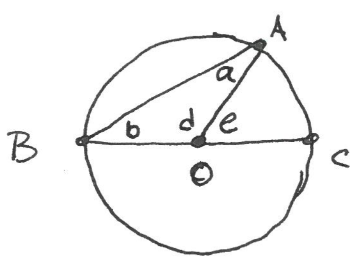
$\overset{\frown}{ACB}$

The inscribed angle theorem says that an inscribed angle measure is $\frac{1}{2}$ of the corresponding arc-angle or central angle:



$$\angle ABC = \frac{1}{2}\angle AOC = \frac{1}{2}\overset{\frown}{AC}$$

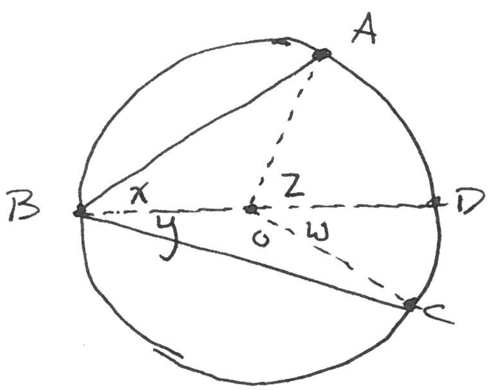
This is easiest to prove for the case where one side of the angle is a diameter

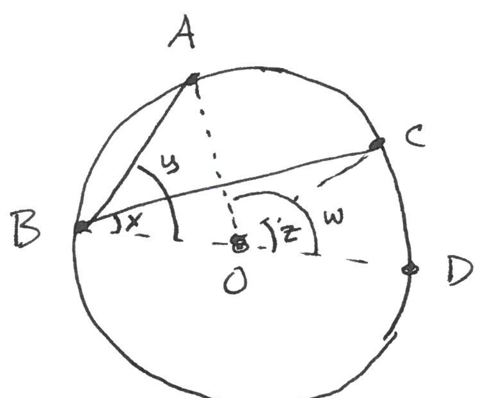


$a+b+d=180$ (Triangle angle sum thm)
$d+e=180°$ (straight line)
$a=b$ (Isosceles triangle thm: $\overline{OA}\cong\overline{OB}$)
→ make substitutions to get
$b=\frac{1}{2}e$.

Inscribed angle theorem

If an inscribed angle does not have a diameter as a side, then the center of the triangle is either inside or outside the center.
In both cases we can add a diameter to the picture and add or subtract angles to get the result





proved case 1

→ $x = \frac{1}{2} z = \frac{1}{2} \widehat{AD}$; $y = \frac{1}{2} w = \frac{1}{2} \widehat{DC}$

so $(x+y) = \frac{1}{2}(z+w)$
$= \frac{1}{2}(\widehat{AC})$

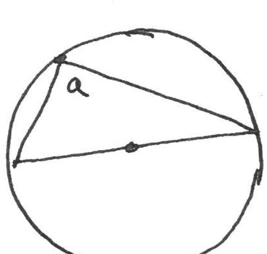
$\angle ABC = \frac{1}{2} \widehat{AC}$

$$\left.\begin{aligned} x &= \tfrac{1}{2} z = \tfrac{1}{2} \widehat{CD} \\ y &= \tfrac{1}{2} w = \tfrac{1}{2} \widehat{AD} \end{aligned}\right\}$$ case 1 (diameter already proved)

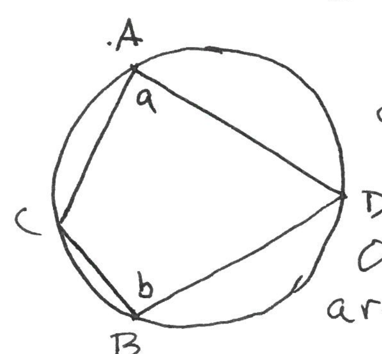
so $y - x = \frac{1}{2}(w - z) = \frac{1}{2} \widehat{AC}$

$\angle ABC = \frac{1}{2} \widehat{AC}$

Cool corollaries:



angle inscribed in semi-circle $a = \frac{1}{2} 180^\circ$
$a = 90^\circ$

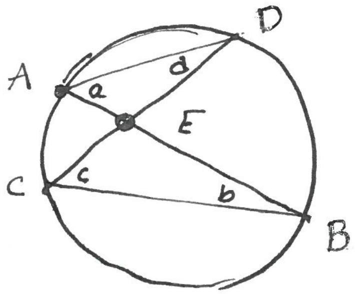


this is called a cyclic quadrilateral.
Opposite angles are supplementary

$a = \frac{1}{2} \widehat{CBD}$ $\quad b = \frac{1}{2} \widehat{CAD}$

$a + b = \frac{1}{2}(\widehat{CBD} + \widehat{CAD}) = \frac{1}{2}(360^\circ)$

$a + b = 180^\circ$



$a = \frac{1}{2}\overset{\frown}{BD}$, $c = \frac{1}{2}\overset{\frown}{BD}$ so $a = c$

$b = \frac{1}{2}\overset{\frown}{AC}$, $d = \frac{1}{2}\overset{\frown}{AC}$ so $b = d$

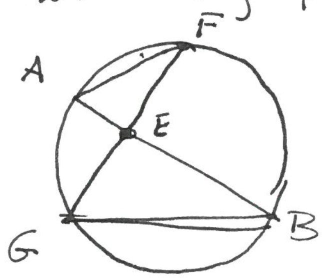
So the triangles are similar (AA similarity thm.)

$\triangle EAD \sim \triangle ECB$

Thus, the sides are proportional: $\frac{EA}{ED} = \frac{EC}{EB}$

$(EB)(EA) = (EC)(ED)$ ☆

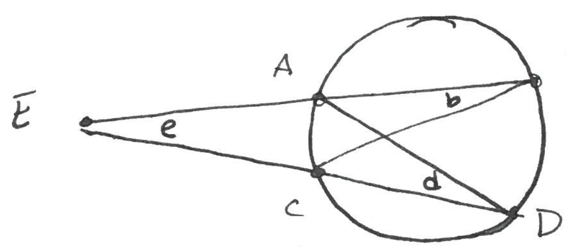
Notice that I could make similar triangles with any pair of lines through E:



$\underbrace{(EA)(EB)} = (EF)(EG) = (EC)(ED)$

this product is the same for this circle and any line through E

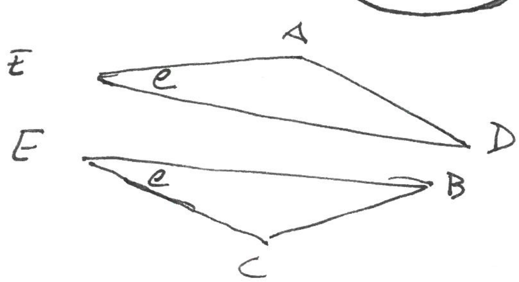
This number is the power of point E with respect to this circle.



$b = \frac{1}{2}\overset{\frown}{AC}$ $d = \frac{1}{2}\overset{\frown}{AC}$ so $b = d$

e is the same angle in both triangles so (AA similarity)

$\triangle EAD \sim \triangle ECB$

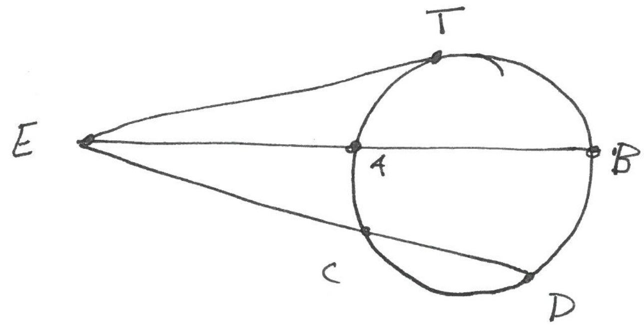


and by using proportional sides, we can get

$(EA)(EB) = (EC)(ED)$

same formula for a point outside a circle

Power of a point: one more trick!



It turns out that if $\overleftrightarrow{ET}$ is tangent to the circle, then

$$\underbrace{(EA)(EB)}_{\text{Power of point } E \text{ with respect to the circle.}} = (ET)(ET)$$

Challenge! the similar triangles here are $\triangle EBT \sim \triangle ETA$. Can you figure out how to prove congruent angles for these?

☆ HW Problem that is not on the handout.

Prove that this construction creates a tangent line (hint: use something from the inscribed angle theorem discussion)

given

Construction:

Let $\mathcal{C} = cir(O, A)$ be a circle with center O that includes point A ($\overline{OA}$ is a radius)

Let E be a point <u>outside</u> the circle.

Construct segment $\overline{OE}$

Construct point M at the midpoint of $\overline{OE}$

Construct circle $\mathcal{D} = cir(M, E)$ with center M, that includes point E.

Let T be one of the intersection points of $\mathcal{C}$ and $\mathcal{D}$.

Prove $\overleftrightarrow{ET}$ is tangent to $\mathcal{C}$.