6.2 # 6, 9

Find a degree 1 or 0 polynomial in each of these cosets of $\mathbb{Q}[x]/((x^2-2))$

a. $x^2+3x+((x^2-2))$

b. $2x^3+x^2+((x^2-2))$

c. $x^4-2x^3+x+3+((x^2-2))$

$\mathbb{Q}[x] = \{ a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \mid a_i \in \mathbb{Q} \}$

$\mathbb{Q}[\sqrt{2}] = \{ b + c\sqrt{2} \mid b, c \in \mathbb{Q} \}$

$f(x), g(x) \in \mathbb{Q}[x]$

Let $a \in \mathbb{R}$

$f(a) \in \mathbb{R}$

$f(x) + g(x) = (f+g)(x)$

$\phi : Q[x] \longrightarrow \mathbb{R}$

$\phi(f(x)) = f(1)$

$\phi(f+g) = (f+g)(1) = f(1) + g(1)$

$= \phi(f) + \phi(g)$

$\phi(f \cdot g) = (f \cdot g)(1) = f(1) \cdot g(1)$

$= \phi(f) + \phi(g)$

ϕ is a homomorphism

$\psi : Q[x] \to \mathbb{Q}[\sqrt{2}]$

$\psi(f) = f(\sqrt{2}) \leftarrow$ function

is it onto?

$a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

$\psi(a + bx) = a + b\sqrt{2}$?

ψ is a homomorphism

$\ker(\psi) = \{ (x^2 - 2) \cdot f(x) \mid f(x) \in \mathbb{Q}[\sqrt{2}] \}$

$= ((x^2 - 2))$

= principal ideal generated by $x^2 - 2$

Put stuff into cosets:

$$\mathbb{Q}[x]/((x^2-2)) =$$

$$\{ f(x) + ((x^2-2)) \mid f(x) \in \mathbb{Q}[x] \}$$

Example

$x^3 + 5 + ((x^2-2))$ is a coset

$$x^3+5 \in x^3+5+((x^2-2))$$

and

$$x^3+5+x^2-2 = x^3+x^2+3 \in x^3+5+((x^2-2))$$

and

$$x^3+5 - x(x^2-2) = x^3+5-x^3+2x = \underbrace{2x+5}_{\uparrow} \in x^3+5+((x^2-2))$$

degree 1 element in the coset.