So here we go. Consider the following grade-school division problem:

$$\begin{array}{ll} \textit{Quotient} \longrightarrow & 11 \\ \textit{Divisor} \longrightarrow & 7\overline{)82} \\ \textit{Dividend} \nearrow & \underline{7} \\ & 12 \\ & \underline{7} \\ \textit{Remainder} \longrightarrow & 5 \end{array} \qquad \text{Check: } \begin{array}{rl} 11 & \longleftarrow \textit{Quotient} \\ \underline{\times 7} & \longleftarrow \textit{Divisor} \\ 77 & \\ \underline{+5} & \longleftarrow \textit{Remainder} \\ 82 & \longleftarrow \textit{Dividend} \end{array}$$

The division process stops when we reach a remainder that is less than the divisor. All the essential facts are contained in the checking procedure, which may be verbally summarized like this:

$$\text{dividend} = (\text{divisor})\,(\text{quotient}) + (\text{remainder}).$$

Here is a formal statement of this idea, in which the dividend is denoted by a, the divisor by b, the quotient by q, and the remainder by r:

Theorem 1.1 The Division Algorithm

divisor

Let a, b be integers with $b > 0$. Then there exist unique integers q and r such that

quotient

Our definition of division

$$a = bq + r \quad \text{and} \quad 0 \le r < b.$$

remainder

Theorem 1.1 allows the possibility that the dividend a might be negative but requires that the remainder r must not only be less than the divisor b but also must be *nonnegative.* To see why this last requirement is necessary, suppose $a = -14$ is divided by $b = 3$, so that $-14 = 3q + r$. If we only require that the remainder be less than the divisor 3, then there are many possibilities for the quotient q and remainder r, including these three:

$$\begin{array}{lll} -14 = 3(-3) + (-5), & \text{with } -5 < 3 & [\textit{Here } q = -3 \textit{ and } r = -5.] \\ -14 = 3(-4) + (-2), & \text{with } -2 < 3 & [\textit{Here } q = -4 \textit{ and } r = -2.] \\ -14 = 3(-5) + 1, & \text{with } 1 < 3 & [\textit{Here } q = -5 \textit{ and } r = 1.]. \end{array}$$

When the remainder is also required to be nonnegative as in Theorem 1.1, then there is exactly one quotient q and one remainder r, namely, $q = -5$ and $r = 1$, as will be shown in the proof.

The fundamental idea underlying the proof of Theorem 1.1 is that division is just repeated subtraction. For example, the division of 82 by 7 is just a shorthand method for repeatedly subtracting 7:

$$\begin{array}{rl} 82 & \\ \underline{-7} & \\ 75 & \longleftarrow 82 - 7 \cdot 1 \\ \underline{-7} & \\ 68 & \longleftarrow 82 - 7 \cdot 2 \\ \underline{-7} & \\ 61 & \longleftarrow 82 - 7 \cdot 3 \\ \underline{-7} & \\ 54 & \longleftarrow 82 - 7 \cdot 4 \\ \underline{-7} & \\ 47 & \longleftarrow 82 - 7 \cdot 5 \\ \underline{-7} & \\ 40 & \longleftarrow 82 - 7 \cdot 6 \end{array} \qquad \begin{array}{rl} 40 & \\ \underline{-7} & \\ 33 & \longleftarrow 82 - 7 \cdot 7 \\ \underline{-7} & \\ 26 & \longleftarrow 82 - 7 \cdot 8 \\ \underline{-7} & \\ 19 & \longleftarrow 82 - 7 \cdot 9 \\ \underline{-7} & \\ 12 & \longleftarrow 82 - 7 \cdot 10 \\ \underline{-7} & \\ 5 & \longleftarrow 82 - 7 \cdot 11 \end{array}$$

The subtractions continue until you reach a nonnegative number less than 7 (in this case 5). The number 5 is the remainder, and the *number* of multiples of 7 that were subtracted (namely, 11, as shown at the right of the subtractions) is the quotient.

In the preceding example we looked at the numbers

$$82 - 7 \cdot 1, \quad 82 - 7 \cdot 2, \quad 82 - 7 \cdot 3, \text{ and so on.}$$

In other words, we looked at numbers of the form $82 - 7x$ for $x = 1, 2, 3, \ldots$ and found the smallest nonnegative one (namely, 5). In the proof of Theorem 1.1 we shall do something very similar.

Examples:

$57 \div 8 = 7\,R\,1$ because $8 \cdot 7 + 1 = 57$ and $0 \le 1 < 8$

$-43 \div 5 = -9\,R\,2$ because $5(-9) + 2 = -43$ and $0 \le 2 < 5$

Practice probs. Find q and r for:

(1) $-17 \div 4$

(2) $-51 \div 6$

(3) $0 \div 19$

(4) $-612 \div 74$