Example: $f: \mathbb{Z}_{12} \to \mathbb{Z}_4$ such that $f([n]_{12}) = [n]_4$

a. To prove: f is a well defined function

Let $a, b \in$ domain such that $a = b$. Prove $f(a) = f(b)$

Let $[n]_{12} = [m]_{12} \in \mathbb{Z}_{12}$

then $n = m + 12p$ for some $p \in \mathbb{Z}$

$f([n]_{12}) = [n]_4$ and $f([m]_{12}) = [m]_4$

$[n]_4 = [m + 12p]_4 = [m]_4 + [12]_4 [p]_4$

$= [m]_4 + 0 = [m]_4$

so $[n]_4 = [m]_4$

so $f([n]_{12}) = f([m]_{12})$

Thus f is a well defined function.

c. To find: $\ker(f)$

Find all $x \in$ domain such that $f(x) = 0$

Let $f([n]_{12}) = [n]_4 = 0$

then $n = 4p$ for some $p \in \mathbb{Z}$

so $\ker(f) = \{[4p]_{12} \mid p \in \mathbb{Z}\} = \{[0]_{12}, [4]_{12}, [8]_{12}\}$

$= 4\mathbb{Z}_{12}$

d. To find: the elements of $\mathbb{Z}_{12}/\ker(f)$

From (c) $\ker(f) = \{[0]_{12}, [4]_{12}, [8]_{12}\} = 4\mathbb{Z}_{12}$

so cosets of $\ker(f)$ are

$[0]_{12} + \ker(f) = \{[0]_{12}, [4]_{12}, [8]_{12}\} = [0]_{12} + 4\mathbb{Z}_{12}$

$[1]_{12} + \ker(f) = \{[1]_{12}, [5]_{12}, [9]_{12}\} = [1]_{12} + 4\mathbb{Z}_{12}$

$[2]_{12} + \ker(f) = \{[2]_{12}, [6]_{12}, [10]_{12}\} = [2]_{12} + 4\mathbb{Z}_{12}$

$[3]_{12} + \ker(f) = \{[3]_{12}, [7]_{12}, [11]_{12}\} = [3]_{12} + 4\mathbb{Z}_{12}$

note that these include all elements of $\mathbb{Z}_{12}$.
any other cosets will be a repetition of one of these

$\mathbb{Z}_{12}/\ker(f) = \{[0]_{12} + \ker(f), [1]_{12} + \ker(f), [2]_{12} + \ker(f), [3]_{12} + \ker(f)\}$

Find a set D of elements in the ring (domain) such that the union of all of the cosets is the whole ring

$$\text{domain } R = \bigcup_{d \in D} d + \ker(f)$$

and the cosets are all distinct:
if $d, e \in D$, then $d + \ker(f) \neq e + \ker(f)$

The answer is all of the cosets $\{d + \ker(f) \mid d \in D\}$

b. To prove: f is a homomorphism

Let $[n]_{12}, [m]_{12} \in \mathbb{Z}_{12}$

$f([n]_{12} + [m]_{12}) = f([n+m]_{12}) = [n+m]_4$

$f([n]_{12}) + f([m]_{12}) = [n]_4 + [m]_4 = [n+m]_4$

so $\underline{f([n]_{12} + [m]_{12}) = f([n]_{12}) + f([m]_{12})}$

$f([n]_{12} \cdot [m]_{12}) = f([n \cdot m]_{12}) = [n \cdot m]_4$

$f([n]_{12}) \cdot f([m]_{12}) = [n]_4 \cdot [m]_4 = [n \cdot m]_4$

so $\underline{f([n]_{12} \cdot [m]_{12}) = f([n]_{12}) \cdot f([m]_{12})}$

Thus f is a homomorphism.

Let a, b ∈ domain
show
$f(a+b) = f(a) + f(b)$
and
$f(ab) = f(a)f(b)$

e. To prove: f is onto

Let $[n]_4 \in \mathbb{Z}_4$

then $[n]_{12} \in \mathbb{Z}_{12}$

and $f([n]_{12}) = [n]_4$

so f is onto

Let a ∈ Codomain
Find b ∈ domain
and show
$f(b) = a$

f. What can you conclude using

$f: \mathbb{Z}_{12} \to \mathbb{Z}_4$ such that $f([n]_{12}) = f([n]_4)$

and the first isomorphism theorem (6.13)?

We know that

$$\mathbb{Z}_{12}/\ker(f) \cong \mathbb{Z}_4$$

is isomorphic to.

g. Define the isomorphism from $\mathbb{Z}_{12}/\ker(f)$ to $\mathbb{Z}_4$

$g: \mathbb{Z}_{12}/\ker(f) \to \mathbb{Z}_4$ such that

$$g([n]_{12} + \ker(f)) = [n]_4$$