Units and Zero Divisors assignment:

1. Given a ring, R, that has a multiplicative identity, prove the product of two units in R is a unit.

Warning: we don't know R is commutative.

Hint: Look at theorem 4, and go look at the video solution to 7.4 #

Second warning: theorem 4 is a useful hint, but this is a problem about rings, not groups, so you can't just say it's true by theorem 4.

Second hint: you can, however, prove it in exactly the same way we proved theorem 4 (just check the inverse conditions)

2. a. Calculate the inverses for the following matrices:

$A=\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$

b. Calculate AB and $A^{-1}B^{-1}$

c. Multiply $(AB)(A^{-1}B^{-1})$. Are these matrices (AB and $A^{-1}B^{-1}$) inverse matrices? If not, what should you do instead to get the inverse of AB (using the technique from problem1 on this assignment)

3. Prove: ***Theorem 43** Any element a of a ring R can't be both a unit and a zero divisor.

Hint: this should be done as a proof by contradiction. Further hints are on the next page if you need them.

4. Prove the statement in 3.2 #21 b: If $a \in R$ is non-zero and is not a zero-divisor, prove that cancellation on the right holds for a. This means that if $ba = ca$ then $b = c$ (see videos for the process for this)

5. Prove that if $1 < a < n$ and $a \mid n$ and $ab \neq 0$ in $\mathbb{Z}_n$, then ab is a zero divisor in $\mathbb{Z}_n$.

Note that the condition $1 < a < n$ and $a \mid n$ and $ab \neq 0$ in $\mathbb{Z}_n$ is equivalent to saying that $1 < \gcd(ab, n) < n$

Hints below/on the next page

6. Using the results you proved in problems 4 and 5, for each equation, tell whether it is safe to cancel the common factor:

a. $2x = 2 \cdot 3$ in $\mathbb{Z}_8$	yes: this implies $x = 3$	no: does not imply $x = 3$ (more solutions possible)
b. $6x = 6 \cdot 3$ in $\mathbb{Z}_8$	yes: this implies $x = 3$	no: does not imply $x = 3$ (more solutions possible)
c. $3x = 3 \cdot 2$ in $\mathbb{Z}_8$	yes: this implies $x = 2$	no: does not imply $x = 2$ (more solutions possible)
d. $2x = 2 \cdot 3$ in $\mathbb{Z}_9$	yes: this implies $x = 3$	no: does not imply $x = 3$ (more solutions possible)
e. $3x = 3 \cdot 2$ in $\mathbb{Z}_9$	yes: this implies $x = 2$	no: does not imply $x = 2$ (more solutions possible)

Hint-through for #3:

- Suppose that a is both a unit and a zero divisor
- Using the zero-divisor definition, name an element to multiply with a and get 0. Be sure to write down the condition that neither factor is equal to 0.
- Using the unit condition, name an element to be the multiplicative inverse of a.
- You now have three different elements. Multiply them together so that a is the middle of the three factors.
- Use the associative law to group on the left, and then multiply: what do you get?
- Use the associative law to group on the right and then multiply: what do you get?
- This means two things are equal: what are they? Explain why that's a contradiction.
- Conclude that a can't be both a unit and a zero divisor.

Hint through for #5:

- $a \mid n$ means that you can write down an equation with a, n, and another integer. Name the third integer, and write down the equation.
- Notice that $n = 0$ in $\mathbb{Z}_n$. Use that to simplify your equation from the previous step as an equation in $\mathbb{Z}_n$
- You named an integer in the first step. What happens when you multiply that integer by ab**?**
- Your previous equation should have proved that ab is a zero divisor in $\mathbb{Z}_n$