Homework on Images, Kernels and Isomorphisms

Theorem 53/78: If R and S are rings, and $f : R \to S$ is a ring homomorphism, then $f(R) = \{f(x) \mid x \in R\} \subseteq S$ is a subring of S.

Proof:

first: show that $f(R)$ *is closed under addition*

Let $f(a),(b) \in f(R)$

then $a,b \in R$

and $f(a)+f(b) = f(a+b)$ because f is a homomorphism,

and $a+b \in R$, so $f(a+b) \in f(R)$

So $f(a)+f(b) \in f(R)$

second: show has $f(R)$ *additive inverses*

Let $f(a) \in R$

then $a \in R$ and therefore $-a \in R$, so $f(-a) \in f(R)$

by theorem 77, we know $f(-a) = -f(a)$, so $-f(a) \in f(R)$

third: show $f(R)$ *is closed under multiplication*

1. Finish the proof of theorem 53/78 by proving the third part:

show $f(R)$ ***is closed under multiplication (you do not need to rewrite parts 1 and 2)***

Theorem 79: If R and S are rings, and $f : R \to S$ is a ring homomorphism, then $\ker(f) \subseteq R$ is an ideal in R.

Proof:

first: prove $\ker(f)$ *is closed under addition*

Let $a,b \in \ker(f)$

then $f(a) = f(b) = 0$

And $f(a+b) = f(a)+f(b)$ because f is a homomorphism

Therefore $f(a+b) = f(a)+f(b) = 0+0 = 0$

so $a+b \in \ker(f)$

second: prove $\ker(f)$ *includes additive inverses*

Let $a \in \ker(f)$

$-a \in R$ and $f(-a) = -f(a)$ by theorem 77

So $f(-a) = -f(a) = -0 = 0$

Therefore $-a \in \ker(f)$

2. Finish the proof of theorem 79 by showing the third part:

prove that $\ker(f)$ ***multiplicatively absorbs elements of R***

(you do not need to rewrite parts 1 and 2)

3. In the proof of theorem 81, why was theorem 58 important? What did it help us to prove?

4. Use Theorem 82 to write down an isomorphism between $\mathbb{Q}[x]/\left(x^2+4\right)$ and a subset of $\mathbb{C}$.