We have the group that is all of the units mod 10 under multiplication:

$U_{10} = \{1, 3, 7, 9\}$

When we look at possible generators, we find that 3 generates the whole group:

$< 3 >= U_{10}$ because

$[3]_{10}{}^{1} \equiv [3]_{10}$ $\quad [3]_{10}{}^{2} \equiv [9]_{10}$ $\quad [3]_{10}{}^{3} \equiv [27]_{10} \equiv [27]_{10}$ $\quad [3]_{10}{}^{4} \equiv [3]_{10}{}^{3} \cdot [3]_{10} \equiv [7 \cdot 3]_{10} \equiv [21]_{10} \equiv [1]_{10}$

Consider (define) $g : U_{10} \to \mathbb{Z}_4$ such that $g([3]_{10}{}^{n}) = [n]_4$

Note: I decided that $[3]_{10}{}^{n}$ is a better notation than $[3^n]_{10}$ because $[3]_{10}{}^{-1} = [7]_{10}$ is easier to make sense of than $[3^{-1}]_{10} = [1/3]_{10}$ =??? But if you write it the other way it's fine.

Lemma: $[3]_{10}{}^{n} \equiv [3]_{10}{}^{m}$ if and only if $m - n = 4k$ where k is an integer.

Part 1 (the tricky one): Let $[3]_{10}{}^{n} \equiv [3]_{10}{}^{m}$

Then $[3]_{10}{}^{n} \cdot [3]_{10}{}^{-m} \equiv [3]_{10}{}^{m} \cdot [3]_{10}{}^{-m}$

So $[3]_{10}{}^{n-m} \equiv 1$

Which means *n*-*m* is a multiple of 4 (because $[3]_{10}{}^{4} \equiv 1$)

Therefore $n - m = 4k$ for some $k \in \mathbb{Z}$

Part 2 (not tricky at all): Let $n - m = 4k$

Then $[3]_{10}{}^{n-m} \equiv [3]_{10}{}^{4k} \equiv ([3]_{10}{}^{4})^{k} \equiv 1$

And $[3]_{10}{}^{n-m}[3]_{10}{}^{m} \equiv 1 \cdot [3]_{10}{}^{m}$

So $[3]_{10}{}^{n} \equiv [3]_{10}{}^{m}$

Part 1: Prove g is a function

Let $[3]_{10}{}^{n}, [3]_{10}{}^{m} \in U_{10}$ such that $[3]_{10}{}^{n} \equiv [3]_{10}{}^{m}$

By the lemma: $n - m = 4k$ for some $k \in \mathbb{Z}$

Thus $[n]_4 \equiv [m]_4$ (by definition of mod congruence)

Therefore $g([3]_{10}{}^{n}) = [n]_4 \equiv [m]_4 = g([3]_{10}{}^{m})$

So g is a function

Part 2: Prove g is one-to-one

Let $[3]_{10}{}^{n}, [3]_{10}{}^{m} \in U_{10}$ such that $g([3]_{10}{}^{n}) = g([3]_{10}{}^{m})$

Then $[n]_4 = [m]_4$

So, by definition of mod number congruence, $n - m = 4k$ for some $k \in \mathbb{Z}$

Thus by the lemma, $[3]_{10}{}^{n} \equiv [3]_{10}{}^{m}$

Therefore g is one-to-one.

Part 3: Prove g is onto

Let $[n]_4 \in \mathbb{Z}_4$

Then $[3]_{10}{}^{n} \in U_{10}$

and $g([3]_{10}{}^{n}) = [n]_4$

so g is onto.

Part 4: Prove g is a homomorphism:

Let $[3]_{10}{}^{n}, [3]_{10}{}^{m} \in U_{10}$

Then $g([3]_{10}{}^{n} \cdot [3]_{10}{}^{m}) = g([3]_{10}{}^{n+m}) = [n + m]_4$

and $g([3]_{10}{}^{n}) + g([3]_{10}{}^{m}) = [n]_4 + [m]_4 = [n + m]_4$

Hence $g([3]_{10}{}^{n} \cdot [3]_{10}{}^{m}) = g([3]_{10}{}^{n}) + g([3]_{10}{}^{m})$

Therefore g is an homomorphism.

Therefore g is an isomorphism