Let $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$. Note that $A, B \in GL(2,\mathbb{R})$ which is a group with group operation matrix multiplication.

Then $<A>=\{A^n \mid n\in\mathbb{Z}\}$ and $<B>=\{B^n \mid n\in\mathbb{Z}\}$

We deduced that $A^n=\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ and $B^n=\begin{bmatrix}1 & 0\\ n & 1\end{bmatrix}$ for all $n\in\mathbb{Z}$ (We did not prove this, but we could use induction to prove this)

Define function $f:<A>\to<B>$ such that $f\left(\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ n & 1\end{bmatrix}$

(This definition tells how to map each element of $<A>$ to a unique element of $<B>$, so it is a function)

To prove: *f* is a homomorphism.

Let $C, D\in<A>$

Then $C=\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}$ and $D=\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}$ for some numbers $r, y\in\mathbb{Z}$

$$f(C)f(D)=f\left(\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}\right)f\left(\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ r & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ y & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ r+y & 0\end{bmatrix}$$

and

$$f(CD)=f\left(\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}\right)=f\left(\begin{bmatrix}1 & r+y\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ r+y & 1\end{bmatrix}$$

Hence $f(C)f(D)=f(CD)$ for all $C, D\in<A>$, **and *f* is a homomorphism.**

To prove: f is 1-to-1.

Let $C, D\in<A>$

Suppose $f(C)=f(D)$

then $f\left(\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}\right)=f\left(\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}\right)$

and $\begin{bmatrix}1 & 0\\ r & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ y & 1\end{bmatrix}$

so $r=y$

therefore $\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}$, which says $C=D$

Thus we have proved that if $f(C)=f(D)$ then $C=D$

Which is equivalent (contrapositive) to saying if $C\neq D$ then $f(C)\neq f(D)$

And *f* is 1-to-1.

Prove *f* is onto:
Let $F \in <B>$
Then $F = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ for some $t \in \mathbb{Z}$
Because $t \in \mathbb{Z}$, we know $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in <A>$
And $f\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = F$
So, for any element $F \in <B>$, there is an element of *A* that is mapped to *F* by *f*.
Hence *f* is onto.

This proves *f* is a 1-to-1, onto homomorphism, so ***f* is an isomorphism,** and $<A> \cong <B>$ ($<A>$ and $<B>$ are isomorphic).