Problem 30: Given a ring *R* with identity. Let $U \subseteq R$ be the set of all units in *R*. Prove that $U, \cdot$ is a group.

a. Show that *U* is closed under multiplication
Let $a, b \in U$, then (by definition) *a*, and *b* are units in *R*, so they have inverses $a^{-1}, b^{-1} \in R$
Consider the element $b^{-1}a^{-1} \in R$
$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = a(1)a^{-1} = aa^{-1} = 1$ a
and
$(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}(1)b = b^{-1}b = 1$
so *ab* has an inverse $b^{-1}a^{-1} \in R$
so, *ab* is a unit in *R* and $ab \in U$

b. Show that *U* has an identity:
We are given that *R* has an identity, $1 \in R$
We know $1 \cdot 1 = 1 \cdot 1 = 1$ so 1 is its own inverse, and hence 1 is a unit.
Thus $1 \in U$

c. Show that elements in *U* have (multiplicative) inverses in *U*:
Let $a \in U$
by definition *a* is a unit in *R*, so it has an inverse $a^{-1} \in R$
by definition of inverse, $a \cdot a^{-1} = a^{-1} \cdot a = 1$
Hence *a* is the inverse of a^{-1}, so a^{-1} is a unit in *R*.
Thus $a^{-1} \in U$

d. Show that multiplication is associative in *U*.
U is a subset of *R*, and multiplication is associative in *R*, so it is associative in *U*.

Thus *U* is a group.