April 8, 2019

Abstract Algebra Problems

1. Find the subgroup of M_2 (addition): $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \right\rangle$

2. Find the subgroup of S_4 $<(1\ 2\ 4)>$

Problem 10: What are the elements of $<a>$ if $a \in G$ and G is a group with binary operation +.

Theorem 11: In a group G, with element $a \in G$, then $<a>$ is a commutative group.

Problem 12: Describe each of these subgroups in a way that lists or describes all of the elements of the subgroup

a) $<4> \subseteq \mathbb{Z}$
b) $<4,6> \subseteq \mathbb{Z}$
c) $<\pi> \subseteq \mathbb{R}$
d) $<(1\ 2\ 3\ 4\ 5)> \subseteq S_5$
e) $<(1\ 2),(3\ 4)> \subseteq S_4$
f) $<(1\ 2),(1\ 2\ 4)> \subseteq S_4$
g) $<r_{90}> \subseteq D_8$ where r_{90} is the $90°$ clockwise rotation.

Problem 13: Which of the subgroups in problem 12 are abelian?

Problem 14: Give an example of a group that is not abelian.

Problem 15: Find a subset of the complex numbers that forms a group with the operation of multiplication. Explain how you know that it is closed and has inverses. Explain why 0 is not in your set.

Problem 16: Find a subset of the 2×2 matrices that is a group with the operation of matrix multiplication. Explain how you know that it is closed and has inverses.

Problem 17: Which of these sets are groups under matrix multiplication? Explain.

a. $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\}$

b. $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a > 0;\ a \in \mathbb{R} \right\}$

c. $S = \left\{ \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} \middle| a,d > 0;\ a,d \in \mathbb{R} \right\}$

d. $S = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \middle| a,b,c > 0;\ a,b,c \in \mathbb{R} \right\}$

April 8, 2019

Problem 18: Prove each of these sets are **not** groups with the identified operation:

a. $\mathbb{R},\times$

b. $S=\left\{\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} \middle| a,d>0;\ a,d\in\mathbb{R}\right\},\times$

c. $\{e,v,u,d\}\subseteq D_3$ the set of reflections and the identity.

d. $S=\left\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \middle| a,b,c>0;\ a,b,c\in\mathbb{R}\right\},\times$

Problem 19: Prove each of these sets **are** groups, with the identified operation:

a. $\mathbb{Q}^*=\left\{\frac{a}{b} \middle| a,b\in\mathbb{Z}, a,b\neq 0\right\},\times$

b. $S=\left\{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \middle| d>0;\ d\in\mathbb{R}\right\},\times$

Problem 20: Prove $a^n a^m = a^{n+m}$ for $|m|>n>0$ and $m<0$ without using induction

Problem 21: Prove $a^n a^m = a^{n+m}$ for $n\geq 0$ and $m<0$ by using induction on *n*.

Problem 22: Let $S=<2>\subseteq\mathbb{Z},+$ which means $S=\{2n\,|\,n\in\mathbb{Z}\}$ (additive)

And let $T=<2>\subseteq\mathbb{R}^*,\times$ which means $T=\{2^n\,|\,n\in\mathbb{Z}\}$ (multiplicative)

Let $f:S\to T$ such that $f(2n)=2^n$

Prove *f* is an isomorphism.

Problem 23: Let $S=<3>\subseteq\mathbb{Z},+$ which means $S=\{3n\,|\,n\in\mathbb{Z}\}$ (additive)

And let $T=<5>\subseteq\mathbb{R}^*,\times$ which means $T=\{5^n\,|\,n\in\mathbb{Z}\}$ (multiplicative)

Let $f:S\to T$ such that $f(3n)=5^n$

Prove *f* is an isomorphism.

Problem 24: Prove that $\mathbb{Z}_n,+$ is a group for any positive integer *n*.

Problem 25: List all of the units and all of the zero divisors of

a. $\mathbb{Z}_4$ b. $\mathbb{Z}_5$ c. $\mathbb{Z}_6$ d. $\mathbb{Z}_7$ e. $\mathbb{Z}_{10}$

April 8, 2019

Problem 26: Let $r \in D_4$ be the counterclockwise rotation by $90°$, and let $S = < r > \subseteq D_4$ be the subgroup generated by *r*. We know from theorem 11 that $S = \{r^n \mid n \in \mathbb{Z}\}, \circ$. You may also assume that you know that r^n is the counterclockwise rotation by $90° \cdot n$

Let $f : S \to \mathbb{Z}_4, +$ such that $f(r^n) = [n]_4$

a. Prove *f* is a function. To do this you must prove that if $r^n = r^m$ then $[n] = [m]$

b. Prove that *f* is a (group) isomorphism.

Problem 27: Given a ring, *R*, and $a, b \in R$ such that $a \neq 0$

a. Does *a* or *R* need any additional properties in order for us to be able to find a solution for *x* that is an element of *R* to the equation $a + x = b$? (If so, what properties do you need? If not, how would you solve it?)

b. Does *a* or *R* need any additional properties in order for us to be able to find a solution for *x* that is an element of *R* to the equation $ax = b$? (If so, what properties do you need? If not, how would you solve it?)

Problem 28:. I would like to have a ring *R* that has the zero-product property, which states:

If $c, d \in R$ and $cd = 0$ then $c = 0$ or $d = 0$

This property will be true if *R* does not have any elements that are ________________________.

Problem 29:. I would like to have a ring *R* where $x = ba^{-1}$ is the solution to the equation $ax = b$ for all $a, b \in R$ such that $a \neq 0$. What additional properties must *R* have in order for this to be true?

Problem 30: Given a ring *R* with identity. Let $U \subseteq R$ be the set of all units in *R*. Prove that $U, \cdot$ is a group.

Problem 31: U_n is the set of units in $\mathbb{Z}_n$. List the elements in:

a) U_5
b) U_8
c) U_{10}
d) U_{12}
e) U_{15}