$S = \langle 2 \rangle \leq \mathbb{Z}_{,+} = \{ 2n \mid n \in \mathbb{Z} \}$

$T = \langle 2 \rangle \leq \mathbb{R}^{+}_{,\times} = \{ 2^n \mid n \in \mathbb{Z} \}$

$f: S \longrightarrow T$

$f(2n) = 2^n$

Let $2i$, $2j \in S$ $\}$ Let $a, b \in S$
then $a = 2i$, $b = 2j$, $i, j \in \mathbb{Z}$

Suppose $f(2i) = f(2j)$

$2^i = 2^j$

$\log_2 2^i = \log_2 2^j$

$i = j$

so $2i = 2j$ (so $a = b$)

so f is 1-to-1

$f: S \to T$

Let $\boxed{2^i \in T}$ (Let $a \in T$ then $a = 2^i$)

then $i \in \mathbb{Z}$

So $\boxed{2i \in S}$

and $f(2i) = 2^i$

so f is onto

Let $a, b \in S$, so $a = 2i$, $b = 2j$ $(i, j \in \mathbb{Z})$

$f(a+b) = f(2i+2j) = f(2(i+j)) = 2^{i+j}$

$f(a) \cdot f(b) = f(2i) \cdot f(2j) = 2^{i} \cdot 2^{j} = 2^{i+j}$

so

$f(a+b) = f(a) \cdot f(b)$ $\Leftrightarrow$ $f(2i+2j) = f(2i) \cdot f(2j)$

so f is a homomorphism

Replacement homework

$S = \langle 3 \rangle \leq \mathbb{Z}, +$, so $S = \{3n \mid n \in \mathbb{Z}\}$

$T = \langle 5 \rangle \leq \mathbb{R}^{+}, \times$, so $T = \{5^{n} \mid n \in \mathbb{Z}\}$

positive reals

$f: S \rightarrow T$ such that $f(3n) = 5^{n}$