Formula to prove:

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Step 1: show that you've checked at least one case

Case n=1: $1^2=1$ and $\frac{1(2\cdot1-1)(2\cdot1+1)}{3}=\frac{1\cdot1\cdot3}{3}=1$

so $1^2=\frac{1(2\cdot1-1)(2\cdot1+1)}{3}$

required

Case n=2: $2\cdot2-1=3$ so the left hand side is $1^2+3^2=1+9=10$

and the right hand side is $\frac{2(2\cdot2-1)(2\cdot2+1)}{3}=\frac{2\cdot3\cdot5}{3}=10$

so $1^2+3^2=\frac{2(2\cdot2-1)(2\cdot2+1)}{3}$

Case n=3: $2\cdot3-1=5$ so the left hand side is $1^2+3^2+5^2=1+9+25=35$

and the right hand side is $\frac{3(2\cdot3-1)(2\cdot3+1)}{3}=\frac{3\cdot5\cdot7}{3}=35$

so $1^2+3^2+5^2=\frac{3(2\cdot3-1)(2\cdot3+1)}{3}$

optional

Step 2: Write the induction hypothesis (assumption)

Assume $1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$ for $1\le n\le k$

which means, in particular that $1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$

Step 3: Copy the left hand side only of the equation on your sticky note. Put in an extra term right before the end of the sum to make your life easier at the next step:

$1^2+3^2+5^2+\ldots+(2(k+1)-1)^2=$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2(k+1)-1)^2=$

Make a substitution using your induction hypothesis:

substitute

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2(k+1)-1)^2=$

$\frac{k(2k-1)(2k+1)}{3}+(2(k+1)-1)^2=$ ← last term doesn't change

Simplify stuff algebraically. If there's a fraction, find a common denominator and add:

$\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2=$

$\frac{k(2k-1)(2k+1)}{3}+\frac{3(2k+1)^2}{3}=$

$\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}=$

Step 2.5: Write on a sticky note or another sheet of paper what you're trying to accomplish next:

$$1^2+3^2+5^2+\ldots+(2(k+1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

Simplify the right hand side, so you'll recognize it when you see it again:

$$\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}=\frac{(k+1)(2k+1)(2k+3)}{3}$$

OK, now look at your sticky note, and if your algebra skills have been properly developed, you should find a way to make your expression look like the thing you want (if that looks hopeless, stick around, I have another strategy coming up for you later in this video)

$$\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}=$$

$$\frac{(k+1)(2k+1)(2k+3)}{3}$$

$$\frac{[k(2k-1)+3(2k+1)](2k+1)}{3}=$$

$$\frac{[2k^2-k+6k+3](2k+1)}{3}=$$

$$\frac{[2k^2+5k+3](2k+1)}{3}=$$

$$\frac{[(2k+3)(k+1)](2k+1)}{3}=$$

$$\frac{(k+1)(2k+1)(2k+3)}{3}=$$

$$\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

So, by the transitive property of equality:

$$1^2+3^2+5^2+\ldots+(2(k+1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

which means the equation must be true for $n=k+1$

So, by induction, $1^2+3^2+5^2+\ldots+(2n-1)^2=\dfrac{n(2n-1)(2n+1)}{3}$ for all $n \geq 1$.

OK. Now I promised another option for doing the last algebraic set of steps (for the algebraically challenged among us—if you work on your factoring skills you won't need option very often):
The other option is to simplify what you have by multiplying as much as possible, and then do the same thing for the right hand side from your sticky note.

$$\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}=$$

$$\frac{k(4k^2+2k-2k-1)+3(4k^2+2k+2k+1)}{3}=$$

$$\frac{k(4k^2-1)+3(4k^2+4k+1)}{3}=$$

$$\frac{4k^3-k+12k^2+12k+3}{3}=$$

$$=\frac{4k^3+12k^2+11k+3}{3}$$

So (using the transitive property of equality)

$$1^2+3^2+5^2+\ldots+(2(k+1)-1)^2=\frac{4k^3+12k^2+11k+3}{3}$$

And (write the right hand side from your sticky note next)

$$\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}=$$

Now simplify it by multiplying out as much as possible:

$$\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}=$$

$$\frac{(k+1)(2k+1)(2k+3)}{3}=$$

$$\frac{(k+1)(4k^2+6k+2k+3)}{3}=$$

$$\frac{(k+1)(4k^2+8k+3)}{3}=$$

$$\frac{4k^3+8k^2+3k+4k^2+8k+3}{3}=$$

$$\frac{4k^3+12k^2+11k+3}{3}$$

So (by the transitive property of equality)

$$\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}=\frac{4k^3+12k^2+11k+3}{3}$$

Then (again by the transitive property of equality)

$$1^2+3^2+5^2+\ldots+(2(k+1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

which means the equation must be true for $n=k+1$

So, by induction, $1^2+3^2+5^2+\ldots+(2n-1)^2=\dfrac{n(2n-1)(2n+1)}{3}$ $n \geq 1$