Discrete math practice problems:

1. Induction proofs:

a. $7+12+\ldots+(5n+2)=\dfrac{n(5n+9)}{2}$

proof:

7 = 7

$\dfrac{1(5\cdot 1+9)}{2}=\dfrac{14}{2}=7$

so, $7=\dfrac{1(5\cdot 1+9)}{2}$, and the formula works for $n=1$

Assume $7+12+\ldots+(5k+2)=\dfrac{k(5k+9)}{2}$

Then (LHS):

$$7+12+\ldots+(5(k+1)+2)=$$
$$7+12+\ldots+(5k+2)+(5(k+1)+2)=$$
$$\frac{k(5k+9)}{2}+(5(k+1)+2)=$$
$$\frac{k(5k+9)}{2}+(5k+7)=$$
$$\frac{k(5k+9)}{2}+\frac{2(5k+7)}{2}=$$
$$\frac{k(5k+9)+2(5k+7)}{2}=$$
$$\frac{5k^2+9k+10k+14}{2}=$$
$$\frac{5k^2+19k+14}{2}$$

And (RHS):

$$\frac{(k+1)(5(k+1)+9)}{2}=\frac{(k+1)(5k+5+9)}{2}=$$
$$\frac{(k+1)(5k+14)}{2}=$$
$$\frac{5k^2+14k+5k+14}{2}=$$
$$\frac{5k^2+19k+14}{2}$$

So $7+12+\ldots+(5(k+1)+2)=\dfrac{5k^2+19k+14}{2}$

Therefore $7+12+\ldots+(5n+2)=\dfrac{n(5n+9)}{2}$ for $n\geq 1$

b. $1+7+\ldots+(2n^2-1)=\dfrac{n(2n-1)(n+2)}{3}$

proof:

1=1

$\dfrac{1(2\cdot 1-1)(1+2)}{3}=\dfrac{1\cdot 1\cdot 3}{3}=1$

So $1=\dfrac{1(2\cdot 1-1)(1+2)}{3}$, and the formula works for $n=1$

Assume: $1+7+\ldots+(2k^2-1)=\dfrac{k(2k-1)(k+2)}{3}$

Then (LHS):

$$1+7+\ldots+(2(k+1)^2-1)=$$
$$1+7+\ldots+(2k^2-1)+(2(k+1)^2-1)=$$
$$\frac{k(2k-1)(k+2)}{3}\quad+(2(k+1)^2-1)=$$
$$\frac{k(2k-1)(k+2)}{3}\quad+\frac{3(2(k+1)^2-1)}{3}=$$
$$\frac{k(2k-1)(k+2)+3(2(k+1)^2-1)}{3}=$$
$$\frac{k(2k^2+4k-k-2)+3(2(k^2+2k+1)-1)}{3}=$$
$$\frac{k(2k^2+3k-2)+3(2k^2+4k+2-1)}{3}=$$
$$\frac{2k^3+3k^2-2k+6k^2+12k+3}{3}=\frac{2k^3+9k^2+10k+3}{3}$$

And (RHS):

$$\frac{(k+1)(2(k+1)-1)((k+1)+2)}{3}=$$
$$\frac{(k+1)(2k+1)(k+3)}{3}=$$
$$\frac{(k+1)(2k^2+6k+k+3)}{3}=$$
$$\frac{(k+1)(2k^2+7k+3)}{3}=$$
$$\frac{2k^3+7k^2+3k+2k^2+7k+3}{3}=\frac{2k^3+9k^2+10k+3}{3}$$

So

$$1+7+\ldots+(2(k+1)^2-1)=\frac{(k+1)(2(k+1)-1)((k+1)+2)}{3}$$

Therefore, $1+7+\ldots+(2n^2-1)=\dfrac{n(2n-1)(n+2)}{3}$ for $n\geq 1$

c.For $S(n): \quad 7+12+...+(5n+2)=\frac{n(5n+9)}{2}$,State and verify $S(3)$.

$S(3)=7+12+17=\frac{3(5\cdot 3+9)}{2}$

$7+12+17=36$

$\frac{3(5\cdot 3+9)}{2}=\frac{3\cdot 24}{2}=\frac{72}{2}=36$

so $S(3)=7+12+17=\frac{3(5\cdot 3+9)}{2}$ is true.

2. From the set: {a, b, c, d, e, f}

a. How many subsets does the set have? 2^6

b. How many 4-element subsets does the set have? $C(6,4)=\frac{6\cdot 5\cdot 4\cdot 3}{4\cdot 3\cdot 2\cdot 1}=15$

3. There are 90 widgets that need to be assembled by 8 workers. What is the smallest number that the most efficient worker (the one who assembles the most widgets) could assemble?
If the widgets were distributed as evenly as possible, each worker would do 11 widgets, and there would be 2 more to do, so the two fastest workers (one of whom is the fastest worker) each assemble **12 widgets.**

4. There are 5 flavors of Jolly Ranchers: Grape, Apple, Watermelon, Cherry and Blue Raspberry
a. If I grab 10 Jolly Ranchers at random out of a bowl, how many different combinations could I get?
This is like the donut problem: $C(10+4,4)=C(14,4)=1001$
b. If I randomly choose 4 Jolly Ranchers, what is the probability that they are all the same flavor? There are 5 flavors, so there are 5 ways to get all the same flavor.
There are 5 choices for each Jolly Rancher, so there are 5^4 ways to choose 5.
Both of these can be considered as order counting, so they are compatible, and the probability is $\frac{5}{5^4}=\frac{1}{5^3}=\frac{1}{125}=.008$
c. If I randomly choose 4 Jolly Ranchers, what is the probability that they are 4 different flavors?

$\frac{5\cdot 4\cdot 3\cdot 2}{5^4}=\frac{24}{125}=.192$

d. If I give Jolly ranchers to 4 people (randomly) what is the probability that the first person gets apple, and the last 2 people get grape? These are 4 people, so they are distinguishable outcomes.
The number of ways to give them to 4 people is 5^4
The number of ways to give them so that the first person gets apple and the last 2 get grape is $1\cdot 5\cdot 1\cdot 1=5$ so the probability is $\frac{5}{5^4}=\frac{1}{5^3}=\frac{1}{125}=.008$
d' (the problem I intended to write): If I give Jolly ranchers to 4 people (randomly) what is the probability that the first person gets apple, **or** the last 2 people get grape?
The number of ways to make sure the first is apple or the last 2 are grape is :

$\underbrace{1\cdot 5\cdot 5\cdot 5}_{\text{first is apple}}+\underbrace{5\cdot 5\cdot 1\cdot 1}_{\text{last 2 grape}}-\underbrace{1\cdot 5\cdot 1\cdot 1}_{\text{both conditions}}=145$ so the probability is $\frac{145}{5^4}=\frac{29}{125}=.232$

e. If I grab some Jolly Ranchers without looking, how many do I need to get to be sure I will have at least 3 of the same flavor?
If got even amounts of all 5 flavors I could get 10 without getting 3 the same. If I get 11, there will be at least one flavor where I will have 3 the same.

5. I have a stack of 15 different Pokemon cards. 7 are water type and 8 are fire type. Assume each has a different number of HP (so they can all be distinguished)

a. In how many ways can I choose 5 cards? $C(15,5)=3003$

b. In how many ways can I choose 3 water type and 2 fire type cards? $C(7,3)\cdot C(8,2)=980$

c. If I choose 5 cards at random, what is the probability that 3 are water type and 2 are fire type? $\frac{980}{3003}$

d. If I put down 5 cards, one at a time, how many orders are there? $5!$

e. If I put down 5 cards in a row, what is the probability that the first card has the highest HP? $\frac{1\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{1}{5}$

f. If I put down 5 cards in a row, what is the probability that they are in order of decreasing HP? $\frac{1}{5!}=\frac{1}{120}$

6. a. How many distinguishable rearrangements are there of the word Massachusetts? There are 13 letters including 2 a's, 4 s's, and 2 t's: $\frac{13!}{2!\cdot 4!\cdot 2!}=64,864,800$

b. What is the probability in a rearrangement of Massachusetts that both of the a's will be together, all of the s's will be together and both of the t's will be together? The letters are: 1m, 2a's, 4 s's, 1c, 1h, 1u, 1e, 2t's, for 8 distinct letters, so there are 8 groupings of letters (all a's, s's, and t's are grouped together), so that makes $8!=40,320$. This way of calculating is only distinguishable arrangements (it does not count permutations of the a's, s's and t's as separate arrangements), so it is compatible with part a, so the probability is $\frac{40,320}{64,864,800}$ alternately: $\frac{8!\cdot 2!\cdot 4!\cdot 2!}{13!}$ is the same.

7. Use the factorial formula to prove that $C(n,3)+C(n,4)=C(n+1,4)$ for $n>4$.

LHS:

$$C(n,3)+C(n,4)=\frac{n!}{(n-3)!3!}+\frac{n!}{(n-4)!4!}$$
$$=\frac{n!\cdot 4}{(n-3)!3!\cdot 4}+\frac{(n-3)n!}{(n-3)(n-4)!4!}$$
$$=\frac{n!\cdot 4}{(n-3)!4!}+\frac{(n-3)n!}{(n-3)!4!}$$
$$=\frac{n!\cdot 4+(n-3)n!}{(n-3)!4!}$$
$$=\frac{n!(4+n-3)}{(n-3)!4!}$$
$$=\frac{n!(n+1)}{(n-3)!4!}$$
$$=\frac{(n+1)!}{(n-3)!4!}$$

notice that $n-3$ is one more than $n-4$ so $(n-3)!=(n-3)(n-4)!$
and 4 is one more than 3 so $4!=4\cdot 3!$

RHS:

$$C(n+1,4)=\frac{(n+1)!}{(n+1-4)!4!}$$
$$=\frac{(n+1)!}{(n-3)!4!}$$

Therefore $C(n,3)+C(n,4)=C(n+1,4)$

(note: $n>4$ is only there so that $C(n,4)$ makes sense…I probably should have made it $n\geq 4$: that would also be OK)