Technical definitions of function, one-to-one, onto and invertible and some proofs:

Our definition of function:
A **function** $f : S \to T$ is a relation such that for each element $s \in S$, there is one and only one corresponding element $f(s) \in T$

Our definition of onto:
A function $f : S \to T$ is **onto** (a surjection) if for every element $t \in T$ there is at least one element $s \in S$ such that $f(s) = t$.

Two definitions of one-to-one:
A function $f : S \to T$ is **one-to-one** (an injection) if for every element $t \in T$ there is no more than one element $s \in S$ such that $f(s) = t$

A function $f : S \to T$ is **one-to-one** (an injection) if whenever $a, b \in S$ such that $f(a) = f(b)$ then $a = b$

1. Why do the two definitions of **one-to-one** mean the same thing?

Theorem: Given $f : R \to S$ and $g : S \to T$ such that f and g are both functions, then $g \circ f : R \to T$ is a function.

proof:
Let $r \in R$ (r represents any element in R)
$g \circ f(r) = g(f(r))$
Because f is a function and $r \in R$, $f(r)$ exists and is one and only one element in S
Because g is a function and $f(r) \in S$, then $g(f(r))$ exists and is one and only one element in T
QED

Theorem: Given $f : R \to S$ and $g : S \to T$ such that f and g are both one-to-one functions, then $g \circ f : R \to T$ is a one-to-one function.

proof (Uses definition 2):
We already know that $g \circ f$ is a function.
Suppose $a, b \in R$ such that $g \circ f(a) = g \circ f(b)$
That means $f(a), f(b) \in S$ such that $g(f(a)) = g(f(b))$
Because g is one-to-one, it must be true that $f(a) = f(b)$
So, now we have $a, b \in R$ such that $f(a) = f(b)$
Because f is one-to-one, it must be true that $a = b$
So we have shown that if $a, b \in R$ such that $g \circ f(a) = g \circ f(b)$ then $a = b$ which (by the definition) means that $g \circ f$ is one-to-one.
QED

Theorem: Given $f: R \to S$ and $g: S \to T$ such that f and g are both onto functions, then $g \circ f: R \to T$ is an onto function.

proof:

Let $t \in T$ (t represents a generic element of the set T. We want to find an $r \in R$ such that $g \circ f(r) = t$)

Because g is one-to-one, there must be at least one element in its pre-image. Let $s \in S$ be an element in the pre-image, so that $g(s) = t$

Because $s \in S$ and f is onto, there must be at least one element in the pre-image of s under f. Let $r \in R$ be an element in the pre-image so that $f(r) = s$

Now $g \circ f(r) = g(f(r)) = g(s) = t$, so $g \circ f(r) = t$

By the definition, $g \circ f$ is onto.

(We found an element $r \in R$ in the pre-image of the arbitrary element $t \in T$ under the function $g \circ f$, so we can say that each element of T has at least one element in its pre-image, and $g \circ f$ is onto.)

QED

Our definition of inverse function:

Two functions $f: S \to T$ and $f^{-1}: T \to S$ are called inverse functions if $f \circ f^{-1}(t) = t$ for every element $t \in T$ and if $f^{-1} \circ f(s) = s$ for every element $s \in S$

Our definition of invertible function:

A function $f: S \to T$ is invertible if it has an inverse (if there exists a function f^{-1} such that f and f^{-1} are inverse functions).

Theorem: Given $f: S \to T$ that is both one-to-one and onto, then f is invertible.

proof:

We will define a function $f^{-1}: T \to S$ as follows:

For $t \in T$, because f is onto, there is at least one element $s \in S$ such that $f(s) = t$. Define $f^{-1}(t) = s$

Because f is one-to-one, there is only one such element, so $f^{-1}(t)$ is one and only one element of S, and f^{-1} is a function.

($f^{-1}(t)$ is an element of the pre-image of t under f. Because there is only one element in the pre-image, f^{-1} is a function)

Now $f \circ f^{-1}(t) = f(s)$ where $f(s) = t$, so $f \circ f^{-1}(t) = t$

And $f^{-1} \circ f(s) = f^{-1}(f(s))$. Now $f(s) \in T$ and $f^{-1}(f(s))$ is the pre-image of $f(s)$, so $f^{-1}(f(s)) = a \in S$ such that $f(a) = f(s)$ (it's the element in S that maps to $f(s)$).

Because f is one-to-one, because $f(a) = f(s)$, we know $a = s$ (there is only one element that maps to $f(s)$, so $s = a$).

This shows $f^{-1}(f(s)) = s$, and hence f and f^{-1} are inverses, so f is invertible.

QED.