Inverse functions and more with composition

Inverse functions:

The typical inverse function definition uses composition to define it: f and g are inverse functions if:

$$f \circ g(x) = x \text{ and } g \circ f(y) = y$$

We're using x and y as the inputs, but that doesn't mean that the inputs have to be real numbers: x and y could be standing for any element of a set, even an ordered pair.

Examples:

$f:\mathbb{R}^2 \to \mathbb{C}$ such that $f(x,y) = x + yi$ and $g:\mathbb{C} \to \mathbb{R}^2$ such that $g(a+bi) = (a,b)$ are inverse functions: $f \circ g(a+bi) = f(a,b) = a+bi$ $g \circ f(x,y) = g(x+yi) = (x,y)$	$f:\mathbb{R} \to [0,\infty)$ such that $f(x) = x^2$ and $g:[0,\infty) \to \mathbb{R}$ such that $g(y) = \sqrt{y}$ are not inverse functions because $g \circ f(-2) = g(4) = 2 \neq -2$

1. In order for the composition $f \circ g(x) = x$ to make sense, what has to be true about domains and codomains?

2. In order for the composition $g \circ f(y) = y$ to make sense, what has to be true about domains and codomains?

3. If $f : D \to C$ is not an onto function, is it possible for f to have an inverse function? Why or why not?

4. If $f : D \to C$ is not one-to-one, is it possible for f to have an inverse function? Why or why not?

5. If $f : D \to C$ is both one-to-one and onto, does f always have an inverse function? Why or why not?

Example:

$f:\mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x,y)=(x,2-y)$ is one-to-one.
$g:\mathbb{R}^2 \to \mathbb{R}^3$ such that $g(a,b)=(a,b,1)$
And $g \circ f:\mathbb{R}^2 \to \mathbb{R}^3$ such that $g \circ f(x,y)=g(x,2-y)=(x,2-y,1)$ is one-to one.

6. If f and g are both one-to-one functions, and the codomain of f is the domain of g
Is $g \circ f$ always one-to one? Why or why not?

Example:

$f:\mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x,y)=(x,2-y)$ is onto.
$g:\mathbb{R}^2 \to \mathbb{R}$ such that $g(a,b)=b$ is onto
And $g \circ f:\mathbb{R}^2 \to \mathbb{R}$ $g \circ f(x,y)=g(x,2-y)=2-y$ is onto.

7. If f and g are both onto, and the codomain of f is the domain of g
Is $g \circ f$ always onto? Why or why not?