Functions: one-to-one, onto, composition

one-to-one:

A function is one-to-one if every element in the range is mapped to by only one element in the domain:

$f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^3$ is one-to one because each real number y has only one pre-image (one number whose cube is y)	$g(x):\mathbb{R}\to\mathbb{R}$ such that $g(x)=x^3-x$ is not one-to-one because $g(0)=0$ and $g(1)=0$ and $g(-1)=0$	$h:C^1(\mathbb{R})\to\mathbb{R}$ such that $h(f)=f(0)$ is a function but not a one-to-one function. Recall $C^1(\mathbb{R})$ is differentiable functions. Different functions can have the same y-intercept.	$k:\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ such that $k(x)=(x,2x)$ is a one-to-one function. There are no elements in $\mathbb{R}\times\mathbb{R}$ that have more than one number in their pre-image (though some have no numbers in their pre-image)

onto:

A function is onto if every element in the codomain is an element is mapped to by something in the domain,

$f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^3$ is onto, because every real number is in the codomain (is mapped to by some number)	$g:\mathbb{R}\to\mathbb{R}$ such that $g(x)=x^2$ is not onto, because there are no real numbers (in the domain) that map to -1. We say: g maps $\mathbb{R}$ onto $\mathbb{R}$	$h:C^1(\mathbb{R})\to\mathbb{R}$ such that $h(f)=f(0)$ is onto. Every real number is mapped to by a function. For example, $c\in\mathbb{R}$ is mapped to by: $f_c(x)=x+c\in C^1(\mathbb{R})$ because $h(f_c)=f_c(0)=c$	$k:\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ such that $k(x)=(x,2x)$ is not an onto function. The element $(1,1)\in\mathbb{R}\times\mathbb{R}$ is not the image of any real number.

composition:

Recall that functions can be composed if it makes sense for the functions

Given functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=3x+1$ and $g:\mathbb{R}\to\mathbb{R}$ such that $g(x)=x^2$, the functions can be composed in either order: $f\circ g(x)=f(g(x))=3(x^2)+1$ And $g\circ f(x)=g(f(x))=(3x+1)^2$	For the functions $h:\mathbb{R}^2\to\mathbb{R}$ such that $h(x,y)=x+y$ and $k:\mathbb{R}^2\to\mathbb{R}^2$ such that $k(x,y)=(2x,-y)$ It would make sense to compose in this order: $h\circ k(x,y)=h(k(x,y))=2x-y$ But it wouldn't work to compose in the opposite order because if you plug (x,y) into h first, you get a real number, not an ordered pair, and you can't use the function k on a real number.

Practice and thinking problems:

1. Fill in the blanks to explain what a one-to-one function and an onto function are:

a. For a one-to-one function, every element in the codomain has ______or_________ points in its pre-image

b. For an onto function, every element in the codomain has _______or_________ points in its pre-image

Functions to use in problems 2 - 4

$f:\mathbb{Z}\to\mathbb{R}$ such that $f(x)=\dfrac{x}{2}$

$g(x)=\mathbb{R}\to\mathbb{R}$ such that $g(x)=\sqrt[3]{x}$

$h:\mathbb{R}^2\to\mathbb{R}^2$ such that $h(x,y)=(y,2x)$

$k:\mathbb{R}^2\to\mathbb{R}$ such that $k(x,y)=\sqrt{x^2+y^2}$

$F:\mathbb{R}\to\mathbb{R}^2$ such that $F(x)=(x,2)$

2. For each of the functions, tell whether it is one-to-one and whether it is onto.

3. For each of these possible function compositions:

- if it makes sense, write the formula for the composition
- if it doesn't make sense, explain why,

a. $f\circ g$

b. $g\circ f$

c. $f\circ h$

d. $h\circ k$

e. $k\circ h$

f. $g\circ k$

g. $k\circ g$

4. Find two examples of a function composition that uses F as one of the two functions.