Math 236 Test 1 review

1. List an element and a subset of each of these sets:

a. $C^1(\mathbb{R})=\{f:\mathbb{R}\to\mathbb{R}\mid f \text{ is differentiable}\}$: $f(x)=x^2\in C^1(\mathbb{R})$ and $\{f(x)=x^2, g(x)=3x+1\}\subseteq C^1(\mathbb{R})$

b. $C^1(\mathbb{R})\times\mathbb{Z}$: $(f(x)=x^2,3)\in C^1\times\mathbb{Z}$ and $\{(f(x)=x^2,3),(g(x)=3x-1,-2)\}\subseteq C^1(\mathbb{R})$

c. $2^{\mathbb{Z}}$: $\{1,2,3\}\in 2^{\mathbb{Z}}$ and $\{\{1,2,3\},\{2,4,6,8,...\}\}\subseteq 2^{\mathbb{Z}}$

d. $E=\{\{(x,y)\in\mathbb{R}^2\mid (ax)^2+(by)^2=1\}\mid a,b\in\mathbb{R}\}$: $\{(x,y)\in\mathbb{R}^2\mid (x)^2+(3y)^2=1\}\in E$ and $\{\{(x,y)\in\mathbb{R}^2\mid (x)^2+(by)^2=1\}\mid b\in\mathbb{R}\}\subseteq E$

2. For each of these statements, tell whether it is true or false. If it is false, tell why.

a. $\{0.5,\sqrt{2}\}\in 2^{\mathbb{R}}$ True, this is a subset of $\mathbb{R}$ so it is an element of the set of all subsets of $\mathbb{R}$

b. $\{0.5,\sqrt{2}\}\in\mathbb{R}\times\mathbb{R}$ False, elements in $\mathbb{R}\times\mathbb{R}$ are ordered pairs, so we would write a pair of numbers with () to show an ordered pair, instead of {} which means a subset.

c. $\{0.5,\sqrt{2}\}\subseteq 2^{\mathbb{R}}$ No, this is an element. It would need another set of {} to be a subset.

d. $\{0.5,\sqrt{2}\}\in 2^{\mathbb{Q}}$ No, $\sqrt{2}$ is irrational, so this is a subset of $\mathbb{R}$ but not a subset of $\mathbb{Q}$

3. Tell a domain and a codomain that would make sense for each function:

a. $f(x,y,z)=(x,y+z)$ $f:\mathbb{R}^3\to\mathbb{R}^2$ b. $g(a+bi)=b$ $g:\mathbb{C}\to\mathbb{R}$

c. $F(f(x),a)=f'(a)$ $F:C^1(\mathbb{R})\times\mathbb{R}\to\mathbb{R}$

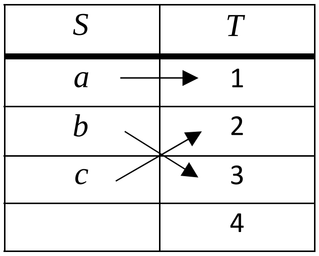
4. For the function $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $f(x,y)=(y,2x)$

Find the images: $f(3,1)=(1,6)$ and $f^{\to}(\{(x,y)\in\mathbb{R}^2\mid y=x^2\})=\{(x,y)\in\mathbb{R}^2\mid 2x=y^2\}$

Find the pre-images: $f^{\leftarrow}(5,2)=(1,5)$ and $f^{\leftarrow}\{(a,b)\in\mathbb{R}^2\mid b=4a+1\}=\{(x,y)\in\mathbb{R}^2\mid 2x=4y+1\}$

5. What has to be true about functions f and g in order for $f\circ g$ to make sense? The codomain of g has to be the same as the domain of f.

6. For each function, tell whether it is one-to-one and whether it is onto, and explain why or why not.

a. $f:S\to T$ such that  There's nothing in T that is mapped to by more than one element, so f **is 1-to-1** The number 4 in T is bit mapped to by anything, so f **is not onto.**	b. $L=\{f(x)=ax+b\mid a,b\in\mathbb{R}\}$ is the set of degree 1 polynomials $H=\{g(x)=c\mid c\in\mathbb{R}\}$ is the set of degree 0 polynomials $F:L\to H$ such that $F(f(x))=f'(x)$ Every constant polynomial $g(x)=c$ is the derivative of something (it has an anti-derivative). For example if $f(x)=cx+1$ then $f'(x)=c=g(x)$. Thus F **is onto.** Every constant polynomial is actually the derivative of more than one thing. For example if $f(x)=cx+1$ and $h(x)=cx+2$ then $f'(x)=h'(x)=c$, so the function F (taking the derivative) is **not one-to-one.**

c. $g:\mathbb{Z}\to\mathbb{Z}_8$ such that $g(n)=[n]_8$

Every element $[n]_8\in\mathbb{Z}_8$ is represented by an integer n, so the function **is onto.**

Several integers correspond to the same element of $\mathbb{Z}_8$, for example $g(2)=[2]_8$ and $g(10)=[10]_8=[2]_8$ so $g(2)=g(10)$ which means g is **not one-to-one**.

d. $h:\mathbb{R}^2\to\mathbb{R}^2$ such that $h(x,y)=(2+x,2^y)$

2^y is always positive, for any real number y, so there are elements of the codomain, such as $(0,-1)$ that are not mapped to by anything in the domain, so h **is not onto.**

You can't have two points mapping to the same place by h, so h **is one-to-one.**

We can check this with algebra: if h was not one-to-one, then there would be two points that map to the same point. I'll name the points (x,y) and (a,b).

$h(x,y)=h(a,b)$ so

$(2+x,2^y)=(2+a,2^b)$,so

$2+x=2+a$ and $2^y=2^b$,

Now we can use algebra on the coordinates separately:

$2+x=2+a$ $\qquad$ $2^y=2^b$

$2+x-2=2+a-2$ and $\log_2(2^y)=\log_2(2^b)$

$x=a$ $\qquad$ $y=b$

So that means (x,y) and (a,b) aren't different points after all, so it has to be one-to-one.

7. What properties need to be satisfied by a function f for it to be invertible? It must be one-to-one and onto.

8. Find the inverse function for each of these functions:

a. $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $f(x,y)=(2y,x+y)$

$f^{-1}:\mathbb{R}^2\to\mathbb{R}^2$ such that

$$f^{-1}(a,b)=\left(b-\frac{a}{2},\frac{a}{2}\right)$$

b. $g:L\to\mathbb{R}^2$ for $L=\{ax+b\mid a,b\in\mathbb{R}\}$

Such that $g(ax+b)=(b,a+b)$

$g^{-1}:\mathbb{R}^2\to L$ such that

$g^{-1}(c,d)=(d-c)x+c$

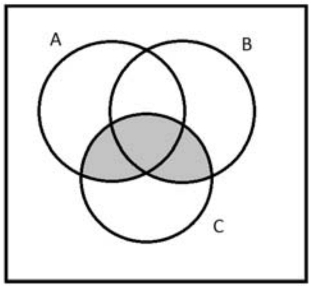
c. $h:\mathbb{R}\to(0,\infty)$ such that $h(x)=2^x$

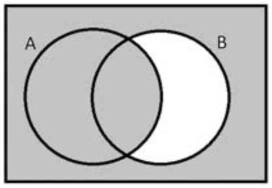
$h^{-1}:(0,\infty)\to\mathbb{R}$ such that

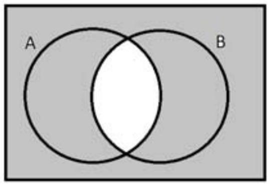
$h^{-1}(y)=\log_2(y)$

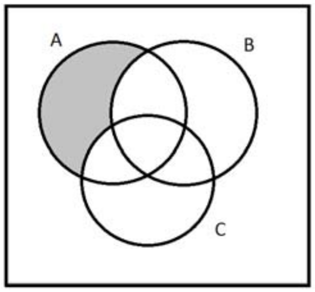
9. Sketch the Venn diagram to show the following set relationships.

a. $(A\cup B)\cap C$ $\qquad$ b. $A\cup\overline{B}$ $\qquad$ c. $\overline{A\cap B}$ $\qquad$ d. $A\cap(\overline{B\cup C})$









10. a and c are equivalent.

10. Make truth tables for each of these, and tell which, if any, are equivalent:

a. $p \to (q \wedge r)$ b. $(\sim q) \wedge (\sim r) \to \sim p$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	p	q	r	$\sim q$	$\sim r$	$(\sim q) \wedge (\sim r)$	$\sim p$	$(\sim q) \wedge (\sim r) \to \sim p$
T	T	T	T	T	T	T	T	F	F	F	F	T
T	T	F	F	F	T	T	F	F	T	F	F	T
T	F	T	F	F	T	F	T	T	F	F	F	T
T	F	F	F	F	T	F	F	T	T	T	F	F
F	T	T	T	T	F	T	T	F	F	F	T	T
F	T	F	F	T	F	T	F	F	T	F	T	T
F	F	T	F	T	F	F	T	T	F	F	T	T
F	F	F	F	T	F	F	F	T	T	T	T	T

c. $\sim (p \wedge (\sim (q \wedge r))$

p	q	r	$q \wedge r$	$\sim (q \wedge r)$	$p \wedge (\sim (q \wedge r)$	$\sim (p \wedge (\sim (q \wedge r))$
T	T	T	T	F	F	T
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	T	T	F
F	T	T	T	F	F	T
F	T	F	F	T	F	T
F	F	T	F	T	F	T
F	F	F	F	T	F	T

a. and c. are equivalent

d. $((\sim q) \vee (\sim r)) \wedge p$

p	q	r	$\sim q$	$\sim r$	$(\sim q) \vee (\sim r)$	$((\sim q) \vee (\sim r)) \wedge p$
T	T	T	F	F	F	F
T	T	F	F	T	T	T
T	F	T	T	F	T	T
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	T	F	F	T	T	F
F	F	T	T	F	T	F
F	F	F	T	T	T	F

11. Write the negation of each statement:

a. Each point in the set lies above the x-axis. **Some point does not lie above the x-axis**

b. No function in the set is a polynomial **Some function in the set is a polynomial**

c. All of the dice rolled the same number. **At least two of the dice rolled different numbers**

d. The numbers in the set are both positive and even. **Some number in the set is not positive or not even.**

e. The numbers in the set are positive or even. **Some number in the set is not positive and not even.**

12. Write the contrapositive of each statement:

a. If a monster is a grue, then it is not happy. **If a monster is happy then it is not a grue.**

b. If a polygon is starlike, then it is both compact and simplicial. **If is polygon is not compact or not simplicial, then it is not starlike.**

c. If a number is constructible or solvable then it is algebraic. **If a number is not algebraic then it is not constructible and not solvable.**

13-14: for each statement, circle the equivalent statements:

13. If it is a square, then it is a rectangle. (2 correct answers)

- **a. All squares are rectangles**
- b. All rectangles are squares
- c. Some squares are rectangles
- **d. All non-rectangles are non-squares**
- e. All non-squares are non-rectangles

14. Every convergent sequence is Cauchy and bounded (2 maybe 3 correct answers)

- a. If a sequence is Cauchy then it is convergent and bounded
- b. If a sequence is bounded, then it is Cauchy and convergent
- **c. If a sequence is convergent then it is Cauchy and bounded**
- **d. If a sequence is not Cauchy or not bounded then it is not convergent**
- e. If a sequence is both not Cauchy and not bounded then it is not convergent
- **f. If a sequence is not both Cauchy and bounded then it is not convergent**

15-18: Prove each statement

15. If k divides n and $a \equiv b \pmod{n}$ then $a \equiv b \pmod{k}$

Given k is a factor of n and $a \equiv b \pmod{n}$

Because k is a factor of n, then $n = kr$

Because $a \equiv b \pmod{n}$, then $a = b + ns$

Substituting the first equation into the second, we get $a = b + (kr)s$

So $a = b + k(rs)$

Which by definition means $a \equiv b \pmod{k}$

16. The sum of an even integer and an odd integer is odd.

Proof:

Given an even integer: $n = 2k$ and an odd integer $m = 2j + 1$

the sum of the integers satisfies:

$$n + m = 2k + 2j + 1$$
$$n + m = 2(k + j) + 1$$

so by definition, $n + m$ is odd.

17. If $a = bc + r$ and d divides both a and b then d divides r

Proof:

because d divides a, then $a = di$

because d divides b, then $b = dj$

We can substitute these into $a = bc + r$ to get

$$di = dj + r$$
$$di - dj = r$$
$$d(i - j) = r$$

so by definition, d divides r

18. If $x + y > 50$ then $x > 10$ or $y > 40$

Proof:

Suppose $x \leq 10$ and $y \leq 40$

then

$$x + y \leq 10 + 40$$
$$x + y \leq 50$$

So if $x + y > 50$ then $x > 10$ or $y > 40$

Definitions and formulas:

$x \equiv y \pmod{n}$ means that $x = y + kn$ (where k is some integer)

An integer n is even means $n = 2k$ for some integer k

An integer n is odd means $n = 2k + 1$ for some integer k

n divides m means that n divides evenly into m, and n is a factor of m. Algebraically it means that $m = nk$ for some integer k