Technical definitions of function, one-to-one, onto and invertible and some proofs:

Our definition of function:
A **function** $f : S \to T$ is a relation such that for each element $s \in S$, there is one and only one corresponding element $f(s) \in T$

Our definition of onto:
A function $f : S \to T$ is **onto** (a surjection) if for every element $t \in T$ there is at least one element $s \in S$ such that $f(s) = t$.

Two definitions of one-to-one:
A function $f : S \to T$ is **one-to-one** (an injection) if for every element $t \in T$ there is no more than one element $s \in S$ such that $f(s) = t$

A function $f : S \to T$ is **one-to-one** (an injection) if whenever $a, b \in S$ such that $f(a) = f(b)$ then $a = b$

1. Why do the two definitions of **one-to-one** mean the same thing?

If f was not one to one, there would be more than one element in S that maps to the same thing in T: $a \neq b$ and $f(a) = f(b)$

For f to be one to one, if is false so then if $f(a) = f(b)$ then $a = b$

2

Theorem: Given $f : R \to S$ and $g : S \to T$ such that f and g are both functions, then $g \circ f : R \to T$ is a function.

proof:
Let $r \in R$ (r represents any element in R)
$g \circ f(r) = g(f(r))$
Because f is a function and $r \in R$, $f(r)$ exists and is one and only one element in S
Because g is a function and $f(r) \in S$, then $g(f(r))$ exists and is one and only one element in T
QED

3

Theorem: Given $f : R \to S$ and $g : S \to T$ such that f and g are both one-to-one functions, then $g \circ f : R \to T$ is a one-to-one function.

proof (Uses definition 2):
We already know that $g \circ f$ is a function.
Suppose $a, b \in R$ such that $g \circ f(a) = g \circ f(b)$
That means $f(a), f(b) \in S$ such that $g(f(a)) = g(f(b))$
Because g is one-to-one, it must be true that $f(a) = f(b)$
So, now we have $a, b \in R$ such that $f(a) = f(b)$
Because f is one-to-one, it must be true that $a = b$
So we have shown that if $a, b \in R$ such that $g \circ f(a) = g \circ f(b)$ then $a = b$ which (by the definition) means that $g \circ f$ is one-to-one.
QED

Theorem: Given $f: R \to S$ and $g: S \to T$ such that f and g are both onto functions, then $g \circ f: R \to T$ is an onto function.

4

proof:

Let $t \in T$ (t represents a generic element of the set T. We want to find an $r \in R$ such that $g \circ f(r) = t$)

Because g is one-to-one, there must be at least one element in its pre-image. Let $s \in S$ be an element in the pre-image, so that $g(s) = t$

Because $s \in S$ and f is onto, there must be at least one element in the pre-image of s under f. Let $r \in R$ be an element in the pre-image so that $f(r) = s$

Now $g \circ f(r) = g(f(r)) = g(s) = t$, so $g \circ f(r) = t$

By the definition, $g \circ f$ is onto.

(We found an element $r \in R$ in the pre-image of the arbitrary element $t \in T$ under the function $g \circ f$, so we can say that each element of T has at least one element in its pre-image, and $g \circ f$ is onto.)

QED

> *Our definition of inverse function:*
> Two functions $f: S \to T$ and $f^{-1}: T \to S$ are called inverse functions if $f \circ f^{-1}(t) = t$ for every element $t \in T$ and if $f^{-1} \circ f(s) = s$ for every element $s \in S$
> *Our definition of invertible function:*
> A function $f: S \to T$ is invertible if it has an inverse (if there exists a function f^{-1} such that f and f^{-1} are inverse functions).

Theorem: Given $f: S \to T$ that is both one-to-one and onto, then f is invertible.

5

proof:

We will define a function $f^{-1}: T \to S$ as follows:

For $t \in T$, because f is onto, there is at least one element $s \in S$ such that $f(s) = t$. Define $f^{-1}(t) = s$

Because f is one-to-one, there is only one such element, so $f^{-1}(t)$ is one and only one element of S, and f^{-1} is a function.

($f^{-1}(t)$ is an element of the pre-image of t under f. Because there is only one element in the pre-image, f^{-1} is a function)

Now $f \circ f^{-1}(t) = f(s)$ where $f(s) = t$, so $f \circ f^{-1}(t) = t$

And $f^{-1} \circ f(s) = f^{-1}(f(s))$. Now $f(s) \in T$ and $f^{-1}(f(s))$ is the pre-image of $f(s)$, so $f^{-1}(f(s)) = a \in S$ such that $f(a) = f(s)$ (it's the element in S that maps to $f(s)$).

Because f is one-to-one, because $f(a) = f(s)$, we know $a = s$ (there is only one element that maps to $f(s)$, so $s = a$).

This shows $f^{-1}(f(s)) = s$, and hence f and f^{-1} are inverses, so f is invertible.

QED.

Assignment: for 5 and either 3 or 4:
Fold a piece of paper in half
On the left copy out the technical proof
On the write, explain what the lines in the technical proof mean in your own words
Example on next page.

(2) Theorem

given func.

$f: R \to S$, $g: S \to T$

then $g \circ s$ is a func.

Let $r \in R$

pick a variable to stand for something in R.

$g \circ f(r) = g(f(r))$

defn. of $g \circ f$

Because f is a func.

$f(r)$ is one & only one element

$$\begin{array}{ccc} R & & S \\ r & \xrightarrow{f} & f(r) \end{array}$$

only one b/c func.

Because g is a func

$g(f(r))$ is one & only one elt.

$$\begin{array}{ccc} S & & T \\ f(r) & \longrightarrow & g(f(r)) \end{array}$$

only one.

so $g \circ f$ is a func.

$r \to f(r) \to g(f(r))$

only one

func. def.

do this ~~for~~

3 or 4

and 5