Finding inverse functions

If a function has an inverse, then you can define the inverse in a really un-helpful way:

$$f^{-1}(y) = x \text{ such that } f(x) = y$$

If f is onto, then we know that there is at least one such x, and if the function is one-to-one then we know there is only one such x, and so f^{-1} exists and is a function.

On the other hand, it would be nice to get a more helpful definition for the inverse function.

Examples:

<table>
<tr>
<td>$f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=(x-1)^3$
For y in the codomain, we want $f^{-1}(y)=x$ such that $f(x)=y$
So, we make the following equation and solve for x:
$(x-1)^3 = y$ $f^{-1}:\mathbb{R}\to\mathbb{R}$ such that
$(x-1)=\sqrt[3]{y}$ so we write $f^{-1}(y)=\sqrt[3]{y}+1$ or
$x=\sqrt[3]{y}+1$ $f^{-1}(x)=\sqrt[3]{x}+1$</td>
<td rowspan="2">Define the following notation:
$\ell_b = \{(x, x+b) \mid x\in\mathbb{R}\}$ will denote the subset of $\mathbb{R}^2$ that is a line with slope 1 and y-intercept b.
Let $L_1 = \{\ell_b \mid b\in\mathbb{R}\}$
Our function is:
$h: L_1\to\mathbb{R}$ such that $h(\ell_b)=4+b$
For y in the codomain, we want
$h^{-1}(y)=\ell_b$ such that $h(\ell_b)=y$
So, make the equation, and solve for b:
$b+4=y$
$b=y-4$
We write the inverse function as:
$h^{-1}:\mathbb{R}\to L_1$ such that
$h^{-1}(y)=\ell_{y-4}$</td>
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<tr>
<td>$g:\mathbb{R}^2\to\mathbb{R}^2$ such that $g(x,y)=(2y,x+3)$
For (a,b) in the codomain, we want $g^{-1}(a,b)=(x,y)$ such that $g(x,y)=(a,b)$
So, we make the following equation and solve for (x,y):
$(2y,x+3)=(a,b)$
$2y=a \Rightarrow y=a/2$
$x+3=b \Rightarrow x=b-3$
So we write the inverse function as:
$g^{-1}:\mathbb{R}^2\to\mathbb{R}^2$ such that
$g^{-1}(a,b)=(b-3,a/2)$ or $g^{-1}(x,y)=(y-3,x/2)$</td>
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if it was $h:(0,1]\to[1,\infty)$ $h(x)=\frac{1}{x}$, then $h^{-1}:[1,\infty)\to(0,1]$
$h^{-1}(x)=\frac{1}{x}$

Practice:

Find inverse functions for:

1. $f:[0,\infty)\to[0,\infty)$ such that $f(x)=\sqrt{x}$ $f^{-1}(y) = y^2$ $f^{-1}: [0,\infty)\to[0,\infty)$
2. $g:\mathbb{R}^2\to\mathbb{R}^2$ such that $g(x,y)=(-y,x+2)$ $g^{-1}(a,b)=(b-2,-a)$ $g^{-1}:\mathbb{R}^2\to\mathbb{R}^2$
3. $h:(0,1]\to[0,\infty)$ such that $h(x)=\frac{1}{x}$ No inverse function because h is not onto (h⁻¹ not defined for $x\in[0,1)$)
   should be $[1,\infty)$

4. Define: $C(O,r)=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=r^2\}$ be the circle with center at the origin and radius r.
Let $Cir=\{C(O,r) \mid r\in(0,\infty)\}$
Find the inverse function of $k:(0,\infty)\to Cir$ such that $k(r)=C(O,r)$

$k^{-1}: Cir\to(0,\infty)$
$k^{-1}(C(O,r))=r$