# Inverse functions and more with composition

**Inverse functions:**

The typical inverse function definition uses composition to define it: $f$ and $g$ are inverse functions if:

$$f \circ g(x) = x \text{ and } g \circ f(y) = y$$

We're using $x$ and $y$ as the inputs, but that doesn't mean that the inputs have to be real numbers: $x$ and $y$ could be standing for any element of a set, even an ordered pair.

**Examples:**

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|---|---|
| $f:\mathbb{R}^2 \to \mathbb{C}$ such that $f(x,y) = x + yi$ and $g:\mathbb{C} \to \mathbb{R}^2$ such that $g(a+bi) = (a,b)$ are inverse functions:<br>$f \circ g(a+bi) = f(a,b) = a+bi$<br>$g \circ f(x,y) = g(x+yi) = (x,y)$ | $f:\mathbb{R} \to [0,\infty)$ such that $f(x) = x^2$ and $g:[0,\infty) \to \mathbb{R}$ such that $g(y) = \sqrt{y}$ are not inverse functions because<br>$g \circ f(-2) = g(4) = 2 \neq -2$ |

1. In order for the composition $f \circ \underline{g(x)} = x$ to make sense, what has to be true about domains and codomains?

codomain of g $\subseteq$ domain of f

2. In order for the composition $g \circ f(y) = y$ to make sense, what has to be true about domains and codomains?

codomain of f $\subseteq$ domain of g

3. If $f: D \to C$ is not an onto function, is it possible for $f$ to have an inverse function? Why or why not?

If it's not onto, there are things in C that are missed by f(D) and they can't map back to D

You can make an inverse function if you change the (restrict) codomain

4. If $f: D \to C$ is not one-to-one, is it possible for $f$ to have an inverse function? Why or why not?

If it's not one-to-one, then some value in the codomain can't decide what to map back to because there's more than one element in the domain that maps to it.

5. If $f: D \to C$ is both one-to-one and onto, does $f$ always have an inverse function? Why or why not?

Yes.

Because every element in the codomain maps back to one and only one element in the domain.

at least one pre-image because f is onto

not more than one pre-image because f is one-to-one

**Technical definitions of function, one-to-one, onto and invertible and some proofs:**

*Our definition of function:*
A **function** $f : S \to T$ is a relation such that for each element $s \in S$, there is one and only one corresponding element $f(s) \in T$

*Our definition of onto:*
A function $f : S \to T$ is **onto** (a surjection) if for every element $t \in T$ there is at least one element $s \in S$ such that $f(s) = t$ .

*Two definitions of one-to-one:*
A function $f : S \to T$ is **one-to-one** (an injection) if for every element $t \in T$ there is no more than one element $s \in S$ such that $f(s) = t$

A function $f : S \to T$ is **one-to-one** (an injection) if whenever $a, b \in S$ such that $f(a) = f(b)$ then $a = b$

1. Why do the two definitions of **one-to-one** mean the same thing?

If f was not one to one, there would be more than one element in S that maps to the same thing in T: $a \neq b$ and $f(a) = f(b)$

For f to be one to one, if is false so then if $f(a) = f(b)$ then $a = b$

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**Theorem**: Given $f : R \to S$ and $g : S \to T$ such that $f$ and $g$ are both functions, then $g \circ f : R \to T$ is a function.

***proof***:
Let $r \in R$ ($r$ represents any element in $R$)
$g \circ f(r) = g(f(r))$
Because $f$ is a function and $r \in R$, $f(r)$ exists and is one and only one element in $S$
Because $g$ is a function and $f(r) \in S$, then $g(f(r))$ exists and is one and only one element in $T$
*QED*

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**Theorem**: Given $f : R \to S$ and $g : S \to T$ such that $f$ and $g$ are both one-to-one functions, then $g \circ f : R \to T$ is a one-to-one function.

***proof*** (Uses definition 2):
We already know that $g \circ f$ is a function.
Suppose $a, b \in R$ such that $g \circ f(a) = g \circ f(b)$
That means $f(a), f(b) \in S$ such that $g(f(a)) = g(f(b))$
Because $g$ is one-to-one, it must be true that $f(a) = f(b)$
So, now we have $a, b \in R$ such that $f(a) = f(b)$
Because $f$ is one-to-one, it must be true that $a = b$
So we have shown that if $a, b \in R$ such that $g \circ f(a) = g \circ f(b)$ then $a = b$ which (by the definition) means that $g \circ f$ is one-to-one.
*QED*