Some problems to practice:
Make a truth table for these. Show each step:

1. $p \vee (\sim p \wedge q)$

| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ | $p \vee (\sim p \wedge q)$ |
|---|---|---|---|---|
| T | T | F | F | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | F | T | F | F |

2. $(p \vee r) \wedge \sim (q \vee r)$

| $p$ | $q$ | $r$ | $p \vee r$ | $q \vee r$ | $\sim (q \vee r)$ | $(p \vee r) \wedge \sim (q \vee r)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F |
| T | T | F | T | T | F | F |
| T | F | T | T | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | T | T | F | F |
| F | T | F | F | T | F | F |
| F | F | T | T | T | F | F |
| F | F | F | F | F | T | F |

3. $\sim p \rightarrow \sim (p \vee q)$

| $p$ | $q$ | $p \vee q$ | $\sim p$ | $\sim (p \vee q)$ | $\sim p \rightarrow \sim (p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | T | F | F | T |
| F | T | T | T | F | F |
| F | F | F | T | T | T |

4. $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \wedge r)$

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ | $p \wedge q$ | $(p \wedge q) \wedge r$ | $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \wedge r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | F |
| T | F | T | T | T | F | F | F |
| T | F | F | F | F | F | F | T |
| F | T | T | T | F | F | F | T |
| F | T | F | T | F | F | F | T |
| F | F | T | T | F | F | F | T |
| F | F | F | F | F | F | F | T |

5. $(p \wedge \sim q) \rightarrow (p \wedge r) \vee \sim (q \wedge r)$

| $p$ | $q$ | $r$ | $\sim q$ | $p \wedge \sim q$ | $p \wedge r$ | $q \wedge r$ | $\sim (q \wedge r)$ | $(p \wedge r) \vee \sim (q \wedge r)$ | $(p \wedge \sim q) \rightarrow (p \wedge r) \vee \sim (q \wedge r)$ |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T | F | T | T |
| T | T | F | F | F | F | F | T | T | T |
| T | F | T | T | T | T | F | T | T | T |
| T | F | F | T | T | F | F | T | T | T |
| F | T | T | F | F | F | T | F | F | T |
| F | T | F | F | F | F | F | T | T | T |
| F | F | T | T | F | F | F | T | T | T |
| F | F | F | T | F | F | F | T | T | T |

Are any of these statements a tautology? #5 is a tautology

Show these statements are logically equivalent:

6. $(p \vee (q \wedge r))$ and $(p \vee r) \wedge (q \vee r)$

| $p$ | $q$ | $r$ | | $q \wedge r$ | $p \vee (q \wedge r)$ | | $p \vee r$ | $q \vee r$ | $(p \vee r) \wedge (q \vee r)$ |
|---|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | | $T$ | $T$ | | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | | $F$ | $T$ | | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | | $F$ | $T$ | | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | | $F$ | $T$ | | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | | $T$ | $T$ | | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | | $F$ | $F$ | | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | | $F$ | $F$ | | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | | $F$ | $F$ | | $F$ | $F$ | $F$ |

***Aargh (typo!) these are not logically equivalent.***

I should have asked for:

$(p \vee (q \wedge r))$ and $(p \vee q) \wedge (p \vee r)$

| $p$ | $q$ | $r$ | | $q \wedge r$ | $p \vee (q \wedge r)$ | | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | | $T$ | $T$ | | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | | $F$ | $T$ | | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | | $F$ | $T$ | | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | | $F$ | $T$ | | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | | $T$ | $T$ | | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | | $F$ | $F$ | | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | | $F$ | $F$ | | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | | $F$ | $F$ | | $F$ | $F$ | $F$ |

Use deMorgan's laws to rewrite these statements:

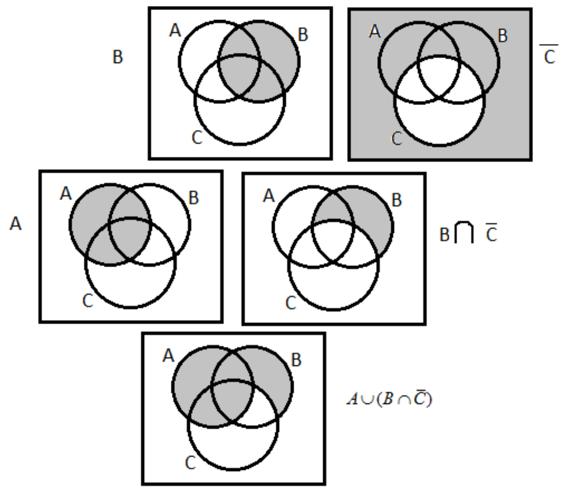
7. $\sim (p \vee q) = \sim p \wedge \sim q$ 8. $\sim p \vee \sim q = \sim (p \wedge q)$

Tell the negation of these statements:

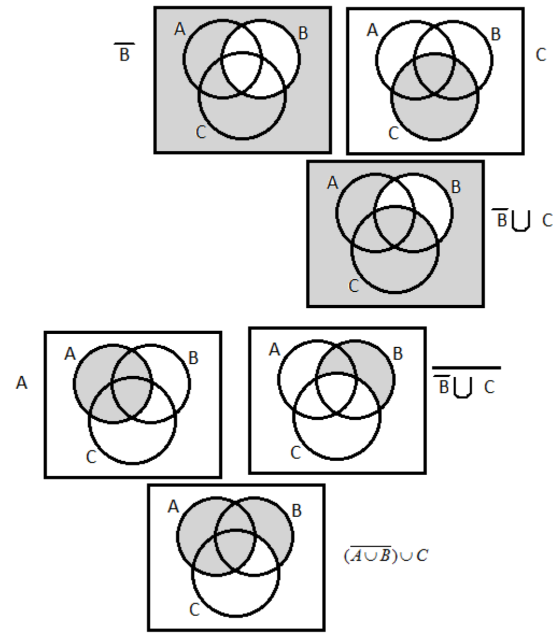
| | |
|---|---|
| 9. The number is prime **and** odd<br>Negation (several versions):<br>• The number is **not both** prime and odd.<br>• The number is **either** not prime **or** not odd.<br>• The number is composite or even.<br>• | 10. **Every** number in set S is either even **or** a multiple of 5.<br>Negation (several versions):<br>• **Some** number in S is both **not** even **and not** a multiple of 5.<br>• **Some** number in S is **neither** even **nor** a multiple of 5<br>• **Some** number in S is **odd and not** a multiple of 5. |

Show a set diagram for each of the following. Show the steps you need to get the final set diagram. Do any of these describe the same set? 11 and 13 describe the same set.

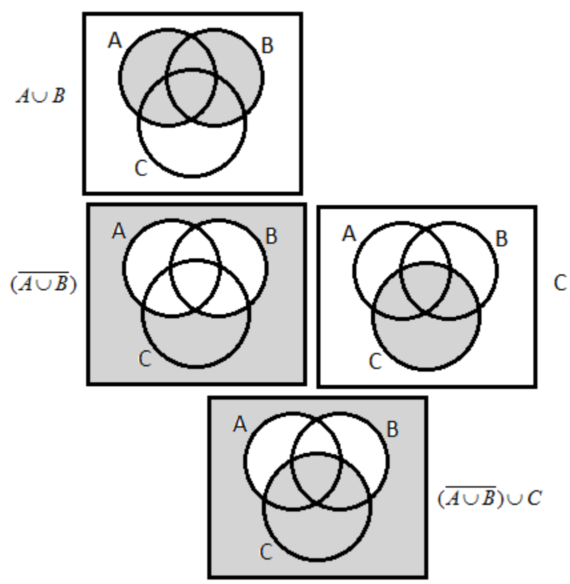
11. $A \cup (B \cap \overline{C})$




13. $A \cup (\overline{\overline{B} \cup C})$




12. $(\overline{A \cup B}) \cup C$



14. Prove that if $a$ is odd and $b$ is odd, then $ab$ is odd.

*For these proofs, you need to know that*

- *You show that a number is even by showing that it is* $2 \bullet \_\_\_$
- *You show that a number is odd by showing that it is* $2 \bullet \_\_\_ + 1$
- *You show that a number is divisible by n by showing that it is* $n \bullet \_\_\_$

*For this proof you are given that two things are odd, so start by writing them in the form of an odd number:*

**Proof:** Since $a$ is odd, $a = 2j + 1$ for some integer $j$

Since $b$ is odd, $b = 2k + 1$ for some integer $k$.

*Then plug in to the expression you want to prove something about: ab*
*Simplify it until you get it in the form you want it:* $2 \bullet \_\_\_ + 1$

$ab = (2j+1)(2k+1)$

$= 4jk + 2j + 2k + 1$

$= 2(2jk + j + k) + 1$

so $ab$ is an odd number

*Finish by writing the conclusion.*

For each of the relations described below, decide if it is reflexive, symmetric or transitive. If it is an equivalence relation, tell how many equivalence classes there are.

15. In the integers, $xRy$ if $y - x = 2$ or $x - y = 2$

This is symmetric (because both orders of $x$ and $y$ do the same thing for making it a relation
but it is not reflexive (1 is not related to 1 because $1 - 1 \neq 2$ )
and it is not transitive. (2 is related to 4 and 4 is related to 6, but 2 is not related to 6)

16. In the integers $xRy$ if $y - x$ is a multiple of 8.

This is symmetric, reflexive and transitive, so it is an equivalence relation (it is the mod 8 equivalence relation). There are 8 equivalence classes, one for each of the numbers: 0, 1, 2, 3, 4, 5, 6, 7.

17. In the set {1, 2, 3, 4, 5, 6}, the relation is given by the ordered pairs:

(1,1), (1,2), (1,3)
(2,1), (2,2), (2,3)
(3,1), (3,2), (3,3)
(4,4), (4,5)
(5, 4), (5,5)
(6,6)

This is reflexive (all of the reflexive relations are listed: (1,1), (2,2), (3,3), (4,4), etc.)
It is symmetric (each relation has a symmetric pair)
It is transitive (it would take a while to check them all)
This is an equivalence relation and has all 3 properties because the set is split into subsets: {1,2,3}, {4,5}, {6} and within each subset all possible relations are listed.
There are 3 equivalence classes (one for each of these subsets)

18. In the real numbers, $xRy$ if $x \leq y$

Reflexive and transitive but not symmetric.

For the equivalence relations below, describe the equivalence class of the given element:
19. In the integers, $xRy$ if $y-x$ is a multiple of 8. Describe the equivalence class containing 6.
Numbers in the equivalence class with 6 satisfy $y-6=8n$ for some integer *n*. Hence, the equivalence class containing 6 contains all numbers of the form $8n+6$ where *n* is an integer.
20. In the real numbers, $xRy$ if the greatest integer function has the same value for both *x* and *y* (the greatest integer function, sometimes called the floor function, is the function that returns the greatest integer that is less than or equal to the number). Describe the equivalence class containing π. Real numbers in the equivalence class are the numbers in the interval $3 \le x < 4$ because they all have greatest integer function = 3.
21. Do the following computations mod 9:
a. $3 \cdot 8+7=24+7=6+7=13=4 (\bmod 9)$
b. $4-6 \cdot 5=4-30=-26+27=1 (\bmod 9)$
c. $6^{17}+5$

$6^1=6$

$6^2=36=0$

$6^{17}=6^{16} \cdot 6=(6^2)^8 \cdot 6=0 \cdot 6=0$

$6^{17}+5=0+5=5$

22. For each of these functions, decide if it is 1-1 and if it is onto:
a. $f(x)=x^2+7$ on the real numbers not 1-1. not onto
b. $y=2^x$ on the real numbers 1-1 but not onto
c. $f(x)=2x$ on the integers 1-1 but not onto for the integers (no odd numbers are in the image)
23. Prove that these functions are 1-1:
a. $f(x)=4x+2$ on the integers
Suppose $f(x)=f(y)$
Then

$4x+2=4y+2$

$\Rightarrow 4x=4y$

$\Rightarrow x=y$

So it is 1-1.
b. $f(x)=x^3$ on the real numbers
Suppose $f(x)=f(y)$
Then

$x^3=y^3$

$\Rightarrow \sqrt[3]{x^3}=\sqrt[3]{y^3}$

$\Rightarrow x=y$

So *f* is 1-1.

24. Prove that these functions are onto:

a. $f(x) = x - 5$ on the integers

Let $y$ be an integer.

Then $y + 5$ is an integer and

$f(y+5) = y + 5 - 5 = y$

So $y$ is in the image of $\mathbb{Z}$ and $f$ is onto.

b. $f(x) = x^3 - x$ on the real numbers.

$f$ is a continuous function on the real numbers (Calculus)

Let $y$ be a real number.

$y$ is positive, negative or 0. $f(0)=0$, so if $y$=0 then it is in the image.

| | |
|---|---|
| If $y$ is positive, choose $b$ to be the larger of 2 and $2y$.<br>Let $a$=0<br>Then $0 = f(a) < y < f(b)$<br>(because if $2y>2$, then $y > 1$ and<br>$(2y)^3 - 2y = 8y^3 - 2y = 2y(8y^2 - 1) > 2y > y$)<br>By the Intermediate value theorem (Calculus) there is a number $c$ between $a$ and $b$ such that $f(c) = y$ | If y is negative choose $a$ to be the smaller of -2 and 2y, and let $b$=0<br>Then $f(a) < f(y) < f(b) = 0$<br>By the Intermediate value theorem (Calculus) there is a number c between a and b such that<br>$f(c) = y$ |

25. Prove that the composition of two 1-1 functions is 1-1.

Proof: Given functions $f : X \to Y$ and $g : Y \to Z$ that are both 1-1 functions.

This means:

| | |
|---|---|
| If $f(a) = f(b)$ for any $a, b \in X$<br>then $a = b$ | If $g(u) = g(v)$ for any $u, v \in Y$<br>then $u = v$ |

There is a function $g \circ f : A \to Z$ (because the codomain of $f$ is the domain of $g$.)

Suppose $g \circ f(a) = g \circ f(b)$ for some $a, b \in X$

Then $g(f(a)) = g(f(b))$ and $g(u) = g(v)$ where $u = f(a)$ and $v = f(b)$

Since $g$ is 1-1, we know that $u = v$

So $f(a) = f(b)$

Since $f$ is 1-1, we know that $a = b$

This if $g \circ f(a) = g \circ f(b)$ for some $a, b \in X$ , then $a = b$ .

This proves that $g \circ f$ is 1-1.

26. Prove that the composition of two onto functions is onto.

Proof: Given functions $f : X \to Y$ and $g : Y \to Z$ that are both onto functions.

This means

| | |
|---|---|
| If $u$ is any element in $Y$, then somewhere in $X$ there is an element $a$ that maps to $u$ (so $f(a) = u$ ) | If $t$ is any element in $Z$, then somewhere in $Y$ there is an element $v$ that maps to $t$ (so $g(v) = t$ ) |

| | |
|---|---|
| There is a function $g \circ f : A \to Z$ | (because the codomain of $f$ is the domain of $g$.) |
| Let $r \in Z$ | (This means: pick any element in $Z$ and name it $r$). |
| Because $g$ is onto, there exists $w \in Y$ such that $g(w) = r$ | This means: somewhere in $Y$ there is an element that maps to $r$; let's name it $w$. |
| Now because $w \in Y$ and $f$ is onto, there exists $b \in X$ such that $f(b) = w$ | This means: somewhere in $X$ there is an element that maps to $w$, let's name it $b$. |
| So, $g(f(b)) = g(w) = r$ | |

We have shown that given any element , there exists and element $b \in X$ such that $g \circ f(b) = r$, and hence $g \circ f$ is onto.