This is the proof by induction that the formula works

$a_n = 3\cdot 3^n + 2(-1)^n$

solves $\underline{a_n = 2a_{n-1} + 3a_{n-2}}$ $a_0 = 5, a_1 = 7$

check! $3\cdot 3^0 + 2(-1)^0 = 5 = a_0$ ✓

$3\cdot 3^1 + 2(-1)^1 = 7$ a_1 ✓

Suppose $a_i = 3\cdot 3^i + 2(-1)^i$

for every $0 \le i \le k$

(That means: $a_k = 3\cdot 3^k + 2(-1)^k$

$a_{k-1} = 3\cdot 3^{k-1} + 2(-1)^{k-1}$))

plug in

$\underline{a_{k+1} = 2a_k + 3a_{k-1}}$

$= 2(3\cdot 3^k + 2(-1)^k) + 3(3\cdot 3^{k-1} + 2(-1)^{k-1})$

$= 2\cdot 3^{k+1} + 4(-1)^k + 3\cdot 3^k + 6(-1)^{k-1}$

$= 2\cdot 3^{k+1} + 4(-1)(-1)^{k-1} + 3^{k+1} + 6(-1)^{k-1}$

$= 3\cdot 3^{k+1} + 2(-1)^{k-1}$

$(-1)^2 = 1$

$= 3\cdot 3^{k+1} + 2(-1)^{k+1}$ ($k+1$ formula!)

so

$a_n = 3\cdot 3^n + 2(-1)^n$ for all $n \ge 0$ □

This is exploring the difference equation by writing out several iterations

$$a_n = 2a_{n-1} + 3a_{n-2} \qquad a_0 = 5,\ a_1 = 7$$

$$a_0 = 5$$

$$a_1 = 7$$

$$a_2 = 2\cdot 7 + 3\cdot 5$$

$$a_3 = 2(2\cdot 7 + 3\cdot 5) + 3\cdot 7$$

$$= 2^2\cdot 7 + 2\cdot 3\cdot 5 + 3\cdot 7$$

$$a_4 = 2(2^2\cdot 7 + 2\cdot 3\cdot 5 + 3\cdot 7) + 3\cdot(2\cdot 7 + 3\cdot 5)$$

$$= 2^3\cdot 7 + 2^2\cdot 3\cdot 5 + 2\cdot 3\cdot 7 + 2\cdot 3\cdot 7 + 3^2\cdot 5$$

$$= 2^3\cdot 7 + 2^2\cdot 3\cdot 5 + \underline{2}\cdot 2\cdot 3\cdot 7 + 3^2\cdot 5$$

$$a_5 = 2(2^3\cdot 7 + 2^2\cdot 3\cdot 5 + \underline{2}\cdot 2\cdot 3\cdot 7 + 3^2\cdot 5) + 3(2^2\cdot 7 + 2\cdot 3\cdot 5 + 3\cdot 7)$$

$$= 2^4\cdot 7 + 2^3\cdot 3\cdot 5 + \underline{2}\cdot 2^2\cdot 3\cdot 7 + 2\cdot 3^2\cdot 5 + 2^2\cdot 3\cdot 7 + 2\cdot 3^2\cdot 5 + 3^2\cdot 7$$

$$= 2^4\cdot 7 + 2^3\cdot 3\cdot 5 + \underline{3}\cdot 2^2\cdot 3\cdot 7 + \underline{2}\cdot 2\cdot 3^2\cdot 5 + 3^2\cdot 7$$

$$a_6 = 2(2^4\cdot 7 + 2^3\cdot 3\cdot 5 + \underline{3}\cdot 2^2\cdot 3\cdot 7 + \underline{2}\cdot 2\cdot 3^2\cdot 5 + 3^2\cdot 7) + 3\cdot(2^3\cdot 7 + 2^2\cdot 3\cdot 5 + \underline{2}\cdot 2\cdot 3\cdot 7 + 3^2\cdot 5)$$

$$= 2^5\cdot 7 + 2^4\cdot 3\cdot 5 + \underline{3}\cdot 2^3\cdot 3\cdot 7 + \underline{2}\cdot 2^2\cdot 3^2\cdot 5 + 2\cdot 3^2\cdot 7 + 2^3\cdot 3\cdot 7 + 2^2\cdot 3^2\cdot 5 + \underline{2}\cdot 2\cdot 3^2\cdot 7 + 3^3\cdot 5$$

$$= 2^5\cdot 7 + 2^4\cdot 3\cdot 5 + \underline{4}\cdot 2^3\cdot 3\cdot 7 + \underline{3}\cdot 2^2\cdot 3^2\cdot 5 + \underline{3}\cdot 2\cdot 3^2\cdot 7 + 3^3\cdot 5$$

$$2^n \qquad 2^{n-1}3 \qquad 2^{n-2}3 \qquad 2^{n-3}3^2 \qquad 2^{n-4}3^2 \qquad 3^3$$

This is deducing some useful facts about solutions to a difference equation if you ignore the initial conditions (a0 and a1)

$\rightarrow \quad a_n = 2a_{n-1} + 3a_{n-2}$

imagine a sequence that does this.

$s_0, s_1, s_2, s_3, s_4, \ldots$

what about

$2s_0, 2s_1, 2s_2, 2s_3, \ldots$?

$2(s_n) = (2 s_{n-1} + 3 s_{n-2})^2$

$2s_n = 2(2s_{n-1}) + 3(2s_{n-2})$

if s & t both work, how about s+t ?

$s_n = 2s_{n-1} + 3s_{n-2}$

$t_n = 2t_{n-1} + 3t_{n-2}$

$(s_n + t_n) = 2(s_{n-1} + t_{n-1}) + 3(s_{n-2} + t_{n-2})$

so if I could find a solution that didnt match a_0 & a_1, I might be able to use it to find the solution I want!

This is investigating r^n as a possible solution to a difference equation (if there were a solution of the form r^n, what would r have to be?)

What about r^n? Could it work w/ the right r?

$r^n = 2r^{n-1} + 3r^{n-2}$

$$\frac{r^n - 2r^{n-1} - 3r^{n-2} = 0}{r^{n-2}}$$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r=3$ & $r=-1$ should work

check it:

$3^n \stackrel{?}{=} 2\cdot 3^{n-1} + 3\cdot 3^{n-2}$

$= 2\cdot 3^{n-1} + 3^{n-1}$

$= 3\cdot 3^{n-1} = 3^n$ ✓

$2(-1)^{n-1} + 3(-1)^{n-2}$

$= (-1)^{n-2}[-2+3]$

$= (-1)^{n-2} = (-1)^n$ ✓

The take away from this page is that if r^n is a solution to a difference equation
a_n=A*a_n-1+B*a_n-2 then r has to be a solution to the quadratic equation
r^2=A*r+B   which is equivalent to r^2-Ar-B=0

Here's how I use my two possible solutions to the difference equation to get a combination of the two that has the initial conditions that I want

Now I've got 3^n & $(-1)^n$ which work for

$a_n = 2a_{n-1} + 3a_{n-2}$.

can I use those to match $a_0 = 5$, $a_1 = 7$?

$3^0 = 1$ $\quad (-1)^0 = 1$ $\quad 3^1 = 3$ $\quad (-1)^1 = -1$

$u \cdot 1 + v \cdot 1 = 5$ $\quad u \cdot 3 + v(-1) = 7$

$$\begin{array}{rl} u \cdot 3 + v(-1) &= 7 \\ u + v &= 5 \\ \hline 4u &= 12 \\ u &= 3 \\ v &= 2. \end{array}$$

Make 2 equations in 2 variables and solve them

My formula is:

$a_n = 3 \cdot 3^n + 2(-1)^n$

this is u*3^n+v*(-1)^n

Let's check: $a_0 = 3 + 2 = 5$

$a_1 = 9 - 2 = 7$

This is a condensed version of the proof by induction on the first page.

if $a_j = 3 \cdot 3^n + 2(-1)^n$ for $j \leq k$

then $a_{k+1} = 2a_k + 3a_{k-1}$

$= 2\left(3 \cdot 3^k + 2(-1)^k\right) + 3\left(3 \cdot 3^{k-1} + 2(-1)^{k-1}\right)$

$= 2 \cdot 3^{k+1} + 4(-1)^k + 3^{k+1} + 6(-1)^{k-1}$

$= 3\, 3^{k+1} + (-1)^{k-1}(-4) + 6(-1)^{k-1} = 3 \cdot 3^{k+1} + 2(-1)^{k+1}$