Chapter 11 practice problems answers

More details:

Sample probs (find a power series expansion for each):

1. $\dfrac{1}{(3+x)^2} = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} n x^{n-1}}{3^{n+1}} = \sum_{m=0}^{\infty} \dfrac{(-1)^m (m+1) x^m}{3^{m+2}}$

2. $\ln(4+x) = \ln(4) + \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+1}}{(n+1)4^{n+1}} = \ln(4) + \sum_{m=1}^{\infty} \dfrac{(-1)^{m-1} x^m}{m 4^m}$

B. Find an infinite series using Taylor's formula:

$f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)(x-a)^n}{n!}$ *Memorize this!*

Sample probs: Find the first 4 non-zero terms of the Taylor series expansion for each function, centered at *a*

3. $\ln(3) + \dfrac{x-3}{3} - \dfrac{(x-3)^2}{18} + \dfrac{(x-3)^3}{81}$

4. $5 + \dfrac{x-25}{10} - \dfrac{(x-25)^2}{1000} + \dfrac{(x-25)^3}{50000}$

5. $1 - 2(x-\pi)^2 + \dfrac{2}{3}(x-\pi)^4 - \dfrac{4}{45}(x-\pi)^6$

6. $e^{x^2} = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{n!}$

7. $x\tan^{-1}(4x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{4^{2n+1} x^{2n+2}}{2n+1}$

D. Find the series to represent an indefinite integral

Sample problem:

8. $\int \sin(x^2)dx = C + \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+3}}{(4n+3)(2n+1)!}$

9. $\int \cos\sqrt{x}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{(n+1)(2n)!}$

II. Do all calculations to at least 4 digits.

9. $\ln(1.5) \approx .4073$ with error .0018

10. $e^3 \approx 16.375$ error 3.7105

B. Estimate a definite integral:

11. $\int_0^1 \sin(x^2)dx \approx .3103$

12. $\int_0^{.5} \ln(1+x^3)\, dx \approx .0151$

13. $\sum_{n=1}^{\infty} \dfrac{4}{n^3} \approx 4.7427$ with the exact answer in the interval $(4.6627, 4.8227)$

14. $\sum_{n=1}^{\infty} \dfrac{6}{n^2} \approx 8.7817$ with the exact answer in the interval (7.5817, 9.9817)

III.

Find the center and radius of convergence for each power series:

15. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{n^2 4^n}$ center 0, radius of convergence = 2

16. $\sum_{n=0}^{\infty} \dfrac{(x-2)^n}{n4^n}$ center 2, radius of convergence 4

17. $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{9^n n!}$ center 0, radius 3

18. $\sum_{n=0}^{\infty} \dfrac{(x+1)^{2n}}{9^n}$ center -1, radius 3

B. Know what an interval of convergence means:

19. A power series $f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n$ has interval of convergence [2, 5). For each of these values of *x*, tell whether you can use the power series to estimate $f(x)$

a. $x = 1$	b. $x = 2$	c. $x = 3$	d. $x = 4$	e. $x = 5$	f. $x = 6$
no	yes	yes	yes	no	no

20. A power series $f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n$ has interval of convergence (-3, 1].

a. tell two values of *x* for which you can use the power series to estimate $f(x)$: x = -2, -1, 0, 1 (any two numbers in the interval)

b. tell two values of *x* for which the power series will not help you estimate $f(x)$: x = -4, -3, 2, 3, 4, (any numbers not in the interval

21. If you wanted to estimate $\int_0^3 \ln(1+x)\, dx$, could you use the power series

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$ $R = 1$

to estimate the integral? Why or why not?

no, because part of the interval (0,3) is not in the interval of convergence (-1,1)

IV. Show whether a series converges or diverges

22. Prove that each of these series converges or diverges using the integral test:

a. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges because

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \int_0^{\infty} u\,du \qquad u = \ln x \quad du = \frac{1}{x}dx$$

$$= \frac{u^2}{2}\Big|_0^{\infty} = \frac{(\ln x)^2}{2}\Big|_1^{\infty} = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^{t}$$

$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2} = \infty$$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{x^3}}$ converges because

$$\int_1^{\infty} \frac{1}{\sqrt{x^3}}\,dx = \ldots = 2$$ (show all steps of integral)

23. Prove that this series converges or diverges using a comparison test:

a. $\sum_{n=1}^{\infty} \frac{(2n+3)}{(n^2+2n+4)}$ diverges by direct comparison test because:

$$\frac{(2n+3)}{(n^2+2n+4)} \geq \frac{n}{(n^2+2n+4)} \geq \frac{n}{(n^2+2n^2+4n^2)}$$

$$= \frac{n}{7n^2} = \frac{1}{7n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{7n} = \frac{1}{7}\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges p-series}$$

OR it diverges by the limit comparison test because:

$$\lim_{n\to\infty} \frac{\frac{(2n+3)}{(n^2+2n+4)}}{1/n} = \lim_{n\to\infty} \frac{(2n+3)n}{(n^2+2n+4)}$$

$$= \lim_{n\to\infty} \frac{(2n^2+3n)\frac{1}{n^2}}{(n^2+2n+4)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2\frac{n^2}{n^2}+3\frac{n}{n^2}}{\frac{n^2}{n^2}+2\frac{n}{n^2}+4\frac{1}{n^2}}$$

$$= \frac{2}{1} = 2 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges p-series}$$

b. $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2n+1}$ converges by direct comparison test because:

$$\frac{3n+1}{n^3+2n+1} \leq \ldots \leq \frac{4}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{4}{n^2} = 4\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges p-series}$$

OR it converges by the limit comparison test because:

$$\lim_{n\to\infty} \frac{\frac{3n+1}{n^3+2n+1}}{1/n^2} = \ldots = 3 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges p-series}$$

(show all steps)

24. has a factorial—use ratio test:

$$\lim_{n\to\infty} \frac{3^{n+1}/(n+1)!}{3^n/n!} = \lim_{n\to\infty} \frac{3^{n+1}n!}{3^n(n+1)!} = \lim_{n\to\infty} \frac{3}{(n+1)} = 0$$

it converges

25. has 3^n try ratio test:

$$\lim_{n\to\infty} \frac{3^{n+1}/(n+1)^2}{3^n/n^2} = \lim_{n\to\infty} \frac{3^{n+1}n^2\frac{1}{n^2}}{3^n(n+1)^2\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3\frac{n^2}{n^2}}{\left((n+1)\frac{1}{n}\right)^2} = \lim_{n\to\infty} \frac{3\cdot 1}{\left(\frac{n}{n}+\frac{1}{n}\right)^2} = \frac{3}{(1+0)^2} = 3$$

it diverges

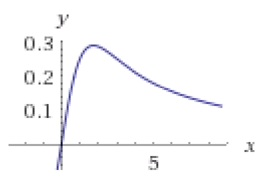
26. alternates—try alternating series test

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$, and $f(x) = \frac{1}{\sqrt{x+1}}$ decreases and the series is alternating, so it converges

27. alternates,

$$\lim_{n\to\infty} \frac{n\frac{1}{n^2}}{(n^2+3)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{3}{n^2}} = \frac{0}{1+0} = 0$$

and $f(x) = \frac{x}{x^2+3}$ decreases when x>2



so it converges.

28. $\sum_{n=1}^{\infty} \frac{(-1)^n n+1}{2n+5}$

$$\lim_{n\to\infty} \frac{(-1)^n n+1}{2n+5} = \lim_{n\to\infty} \frac{\left((-1)^n n+1\right)\frac{1}{n}}{(2n+5)\frac{1}{n}}$$ does not converge

$$= \lim_{n\to\infty} \frac{(-1)^n\frac{n}{n}+\frac{1}{n}}{2\frac{n}{n}+5\frac{1}{n}} = \lim_{n\to\infty} \frac{(-1)^n+0}{2+0}$$

(doesn't converge to 0, and doesn't converge at all) so the sum diverges.

29. Converges by

Direct comparison test

$$\frac{n^2+5}{n^4+3n^2+2} \leq \frac{6}{n^2}$$

OR Limit comparison test

$$\lim_{n\to\infty} \frac{\frac{n^2+5}{n^4+3n^2+2}}{\frac{1}{n^2}} = \ldots = 1$$

30. Converges by ratio test (ratio = 1/4)

Note: problems with a ★ have a complete solution shown, and the others have a short summary of the answer