Graphing a rational function using properties of factors.

To begin, you need a rational function that is in factored form. If you have something that is not factored, your first step is to factor it. I'm going to do an example that is already in factored form (this one would be too complicated to factor quickly)

$$y=\frac{(x-3)^2(x+2)(x+1)}{(x-1)(x+1)(x+4)^2}$$

The first thing to look for is: are there factors that can cancel out? In this case there is a factor of x+1 in both numerator and denominator, so we can simplify the rational function. When we simplify, it leaves a missing point at x=-1 because that value isn't in the domain of the function:

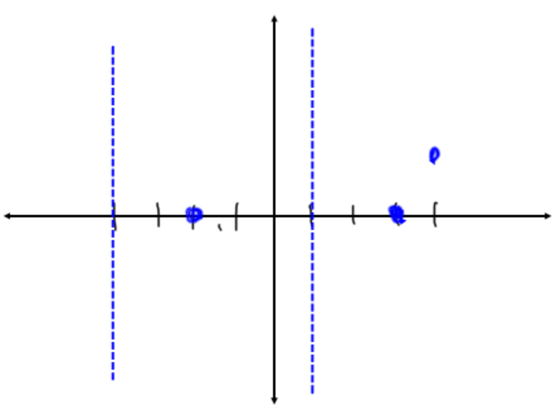
$$y=\frac{(x-3)^2(x+2)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}(x+4)^2}=\frac{(x-3)^2(x+2)}{(x-1)(x+4)^2}\qquad x\neq -1$$

Now, factors in the numerator correspond to zeros (also known as roots) of the function in the graph. This means there are zeros at $x=3,-2$.

Factors in the denominator correspond to vertical asymptotes in the graph. This means there are vertical asymptotes at $x=1,-4$

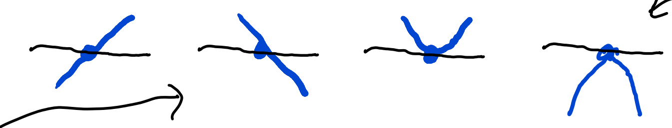
You always need an approximate location of one non-root point to start your graph. If you put in a number bigger than 3, you should get out a positive result, so I'm going to plot an approximate point to the right of 3 and above the x-axis.

So far the graph looks like this



Now, from that approximate point, start moving right or, in this case, left toward the nearest root or asymptote.

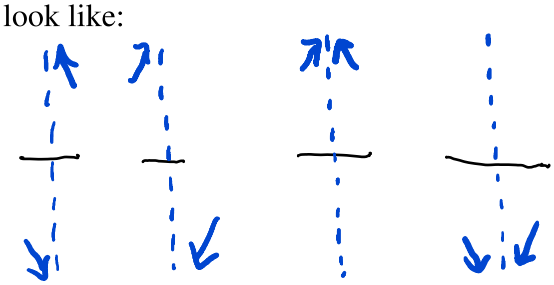
There are only 4 things a root can look like:



The first two correspond to factors with an odd exponent, like $(x+2)$, $(x-5)^3$ or $(2x-1)^9$

The second two correspond to factors with an even exponent like $(x-3)^2$

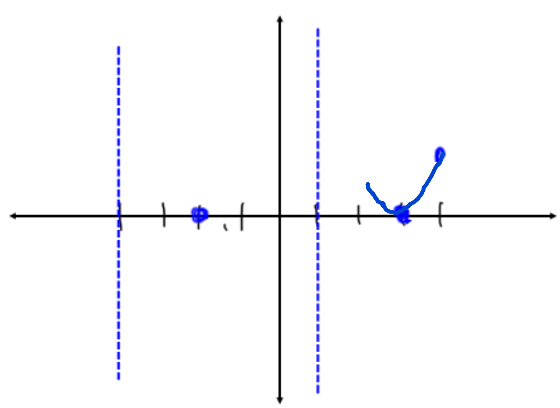
And there are only 4 things a vertical asymptote can look like:



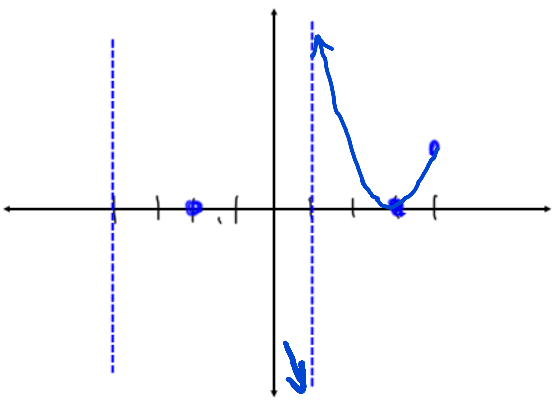
The two on the left correspond to factors with odd exponents (the sign changes) and the two on the right correspond to factors with even exponents (the sign stays the same).

Now, the task is to figure out which pictures belong where. Moving from our approximate point on the right, the factor corresponding to the root at x=3 is $(x-3)^2$, and because the graph has to go up on the right, the picture we need is 

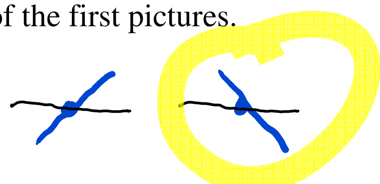
So the graph looks like:



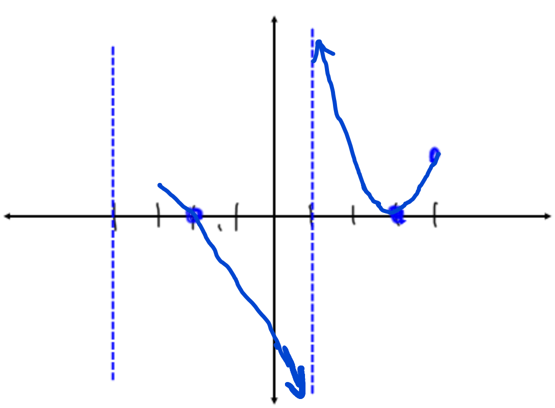
Next to the left, we have an asymptote. Our graph is above the x-axis, so we know it has to go up to the right of the vertical asymptote. The factor that corresponds to the vertical asymptote at x=1 is x-1. It has an odd exponent, so that means that the sign changes and the arrow at the other side of the vertical asymptote has to point down:



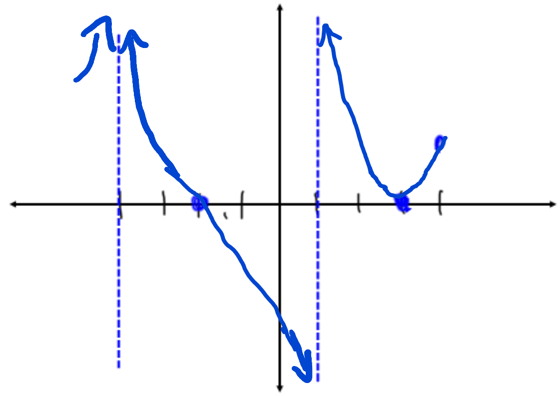
Moving left again, the next interesting thing is the root at x= -2. It corresponds to the factor . It has an odd exponent (1), so that means it has to look like one of the first pictures.



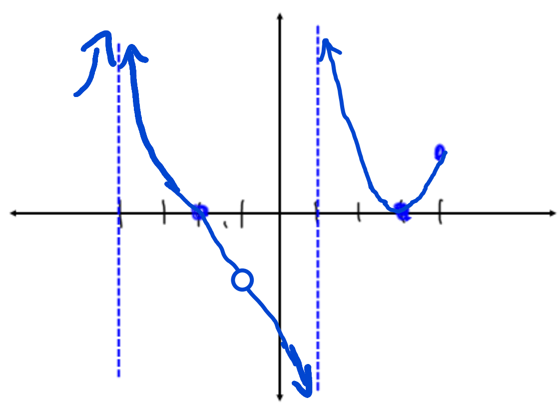
You can see from the asymptote, that it's going to have to be negative to the right of the root, so that tells us it's going to be the second picture. We add that to the graph to get:



The last interesting thing on our graph is the vertical asymptote at x= -4. It corresponds to the factor $(x+4)^2$. Since that has an even exponent, we know both arrows go the same direction (both up or both down). We can see from the root, that it will have to be positive to the right of the asymptote, so that means it has to go up (and in) on both sides of the asymptote:



Finally, we go back and remember to fix up the domain--there's a point missing at $x=-1$--so wherever the graph would otherwise be, there's now a little hole. Draw a little circle to represent the hole.



So, what you get is a graph that shows all of the most interesting features of the function: the roots and the vertical asymptotes. These problems are really good practice for thinking in a calculus way because they get you thinking about how graphs and equations fit together. Your goal in graphing this way isn't to try and get the most accurate numbers (that's what computers are for)--your goal is to associate the components of the function, with their effects on the graph: that's what people do best.